THE STRUCTURE OF THE CATEGORY OF PARABOLIC EQUATIONS [1]

Marina Prokhorova [2]
pmf@imm.uran.ru

Abstract. We define here the category of partial differential equations. Special cases of morphisms from an object (equation) are symmetries of the equation and reductions of the equation by a symmetry groups, but there are many other morphisms. We are mostly interested in a subcategory that arises from second order parabolic equations on arbitrary manifolds. We introduce a certain structure in this category enabling us to find the simplest representative of every quotient object of the given object, and develop a special-purpose language for description and study of structures of this kind. An example that deals with nonlinear reaction-diffusion equation is discussed in more detail.

Key words: category of partial differential equations, factorization of differential equations; parabolic equation; reaction-diffusion equation; heat equation; symmetry group.

Mathematics Subject Classification: 18B99, 35K05, 35K55, 35K57, 58J70

1 Introduction

We define here the category $\mathcal{PDE}$ of partial differential equations, develop a special-purpose language for description and study of its internal structure, and investigate in detail its full subcategory that arises from second order parabolic equations on arbitrary manifolds.

$$\begin{tikzcd} N \arrow[r, "F"] \arrow[d, "\pi"'] & N' \arrow[d, "\pi'"] \\ M \arrow[r] & M' \end{tikzcd}$$

Figure 1:

Let $\pi\colon N \to M$ be a smooth fiber bundle, E be a subset of k-jet bundle $J^k(\pi)$. E could be considered as k-th order partial differential equation for a sections of π, that is $s \in \Gamma\pi$ is the solution of E iff its k-th prolongation $j^k(s) \in \Gamma\left(J^k(\pi) \to M\right)$ is contained in E. Here $\Gamma\pi$ is the space of smooth sections of π.

Suppose $\pi\colon N \to M$, $\pi'\colon N' \to M'$ are smooth fiber bundles, $F\colon \pi \to \pi'$ is a bundle morphism which is diffeomorphism on the fibers (see Fig. 1).

$$\begin{tikzcd} J^k_F(\pi) \arrow[d, hook] \arrow[dr, "F^k"] & \\ J^k(\pi) \arrow[d] & J^k(\pi') \arrow[d] \\ N \arrow[r, two heads, "F"] \arrow[d, "\pi"'] & N' \arrow[d, "\pi'"] \\ M \arrow[r, two heads] & M' \end{tikzcd}$$

Figure 2:

F induces the map $F^*\colon \Gamma\pi' \to \Gamma\pi$; denote by $\Gamma_F\pi$ its image. We say that a section of π is F-projected if it is contained in $\Gamma_F\pi$. If F is surjective then F^* is injective, so we have the map $\Gamma_F\pi \to \Gamma\pi'$. If F is surjective submersion then we have the map $F^k\colon J^k_F(\pi) \to J^k(\pi')$, where $J^k_F(\pi) = F^*J^k(\pi')$ is the bundle of k-jets of F-projected sections of π (see Fig. 2).

Let us define now the category $\mathcal{PDE}_0$. Its objects are pairs $\left(\pi\colon N \to M, E \subset J^k(\pi)\right)$, $k \in \mathbb{N}$, and morphisms from the object $\left(\pi\colon N \to M, E \subset J^k(\pi)\right)$ to the object $\left(\pi'\colon N' \to M', E' \subset J^k(\pi')\right)$ are smooth bundle morphisms $F\colon \pi \to \pi'$ satisfying the following conditions:

1. F define surjective submersion $M \to M'$,

[1] The work was partially supported by a RFBR grant 09-01-00139-a (Russia), and by the Program for Basic Research of Mathematical Sciences Branch of Russian Academy of Sciences. This paper was partially written during my stay at IHES; I am grateful to this institution for hospitality and for the excellent working conditions. I am grateful to Yuri Zarhin and Vladimir Rubtsov for useful remarks.

[2] Institute of Mathematics and Mechanics, Russian Academy of Sciences / Ural Branch.

2. the diagram Fig. 1 is the pullback square in the category of smooth manifolds, that is for any $x \in M$ the map $\pi^{-1}(x) \to \pi'^{-1}(Fx)$ is diffeomorphism,

3. $E \cap J^k_F(\pi) = \left(F^k\right)^{-1}(E')$.

If $F\colon (\pi, E) \to (\pi', E')$ is morphism in $\mathcal{PDE}_0$ then F^* define the bijection between the set of all solutions of E' and the set of all F-projected solutions of E.

In Section 3 we define extended category $\mathcal{PDE}$, whose objects are pairs (N, E) where N is a smooth manifold and E is a subset of the bundle $J^k_m(N)$ of k-jets of m-dimensional submanifolds of N, and whose morphisms from (N, E) to (N', E') are maps $N \to N'$ sutisfying some analogue of conditions (1-3) above. By definition, the solutions of the equation E are smooth m-dimensional non-vertical integral manifolds of the Cartan distribution on $J^k_m(N)$ contained in E. Particularly, the set of solutions includes all m-dimensional submanifolds $L \subset N$ such that k-th prolongation $j^k(L) \subset J^k_m(N)$ is contained in E. Any morphism $F\colon (N, E) \to (N', E')$ of $\mathcal{PDE}$ define the bijection between the set of all solutions of E' and the set of all F-projected solutions of E in the same manner as for $\mathcal{PDE}_0$.

The category $\mathcal{PDE}$ generalize the notion of a symmetry group in two directions:

1. Automorphisms group of the object (N, E) in $\mathcal{PDE}$ is the symmetry group of the equation E.

2. For a symmetry group G of E the natural projection $N \to N/G$ define the morphism $(N, E) \to (N/G, E/G)$ in $\mathcal{PDE}$ where E/G is the equation describing G-invariant solutions of E.

Note that the morphisms of $\mathcal{PDE}$ go beyond the morphisms of this kind.

Further we want to introduce a certain structure in $\mathcal{PDE}$ formed by a lattice of subcategories. We receive these subcategories restricting to the equations of specific kind (for example, elliptic, parabolic, hyperbolic, linear, quasilinear equations etc.) or to the morphisms of specific kind (for example, morphisms respecting the projection of N on the base manifold M as in $\mathcal{PDE}_0$) or both.

When we interested in the solutions of some equation it is useful to look for its quotient objects because every quotient object give us the class of solutions of the original equation. It may happen that the position of an object in the lattice gives information on the morphisms from the object and/or on the kind of the simplest representatives of quotient objects. In Section 5 we develop a special-purpose language for description and study of such situations. We introduce a number of partial orders on the class of all subcategories of fixed category and depict these orders by various arrows (see Table 1 and Fig. 3). For instance, we say that a subcategory $\mathcal{C}_1$ is closed in a category $\mathcal{C}$ and depict $\mathcal{C} \bullet\!\!\longrightarrow \mathcal{C}_1$ if every morphism in $\mathcal{C}$ with source from $\mathcal{C}_1$ is a morphism in $\mathcal{C}_1$; we say that $\mathcal{C}_1$ is plentiful in $\mathcal{C}$ and depict $\mathcal{C} \dashrightarrow \mathcal{C}_1$ if for every $\mathbf{A} \in \mathrm{Ob}_{\mathcal{C}_1}$ and for every quotient object of $\mathbf{A}$ in $\mathcal{C}$ there exist a representative of this quotient object in $\mathcal{C}_1$; and so on.

We use this language for the detail study of the full subcategory $\mathcal{PE}$ of $\mathcal{PDE}$ that arises from second order parabolic equations posed on arbitrary manifolds, but we hope that our approach based on category theory may be useful for other types of PDE as well. Objects of $\mathcal{PE}$ are equations having the form $u_t = \sum_{i,j} b^{ij}(t, x, u)u_{ij} + \sum_{i,j} c^{ij}(t, x, u)u_i u_j + \sum_i b^i(t, x, u)u_i + q(t, x, u)$ in a local coordinates (x^i) on X, X is arbitrary smooth manifold. We prove that every morphism in $\mathcal{PE}$ is of the form $(t, x, u) \mapsto (t'(t), x'(t, x), u'(t, x, u))$.

Particularly, $\mathcal{PE}$ appear to be the subcategory of $\mathcal{PDE}_0$. In Section 7 we investigate the structure of $\mathcal{PE}$; the part of obtained results is shown on Fig. 6, where arrows designate specific partial orders on subcategories as was described above. On every of three diagrams on Fig. 6 down movement mean the restriction of class of permitted equations and right movement mean the restriction of class of permitted morphisms.

Using of the developed structure is illustrated in Section 8 on the example of reaction-diffusion equation $u_t = a(u)\,(\Delta u + \eta\nabla u) + q(x,u)$, $x \in X$, posed on a Riemann manifold X equipped with vector field η; let $\mathbf{A}$ be the corresponding object of $\mathcal{PE}$. If function $a(u)$ is nonlinear enough then we receive the following results as immediate corollary of developed investigation of $\mathcal{PE}$ structure:

- For every morphism $F\colon \mathbf{A} \to \mathbf{A}'$ in $\mathcal{PE}$ there exist bijective change of variables in the quotient equation $\mathbf{A}'$ transforming F to the morphism of the form $(t,x,u) \to (t, x'(x), u)$ and transforming $\mathbf{A}'$ to the equation of the form $u'_t = a(u')(\Delta u' + \eta'\nabla u') + q'(x',u')$, $x' \in X'$, posed on the Riemann manifold X' equipped with vector field η', with the same function a.

- If a morphism $F\colon \mathbf{A} \to \mathbf{A}'$ of the category $\mathcal{PE}$ has the form $(t,x,u) \mapsto (t, x'(t,x), u'(t,x,u))$ and $\mathbf{A}'$ is of the form $u'_t = a(u')(\Delta u' + \eta'\nabla u') + q'(x',u')$, then F is of the form $(t,x,u) \to (t,x'(x),u)$.

In Sections 3 and 9 we discuss the relations between our approach to the factorization of PDE and the other approaches.

Section 10 contains proofs of the theorems.

2 The category $\mathcal{PDE}_0$ of partial differential equations

Let M, K be smooth manifolds. A system E of k-th order partial differential equations for a function $u\colon M \to K$ is given as a system of equations $\Phi^l(x, u, \dots, u^{(k)}) = 0$ involving x, u and the derivatives of u with respect to x up to order k, where $x = (x^1, \dots, x^m)$ are local coordinates on M and $u = (u^1, \dots, u^j)$ are local coordinates on K. Further we will use a words "partial differential equation", "PDE" or "equation" instead of "a partial differential equation or a system of partial differential equations" for short.

Remember some things about jets and related notions. k-jet of a smooth function $u\colon M \to K$ at the point $x \in M$ is the equivalence class of smooth functions $M \to K$ whose value and partial derivatives up to k-th order at x coincide with the ones of u. All k-jets of all smooth functions $M \to K$ form the smooth manifold $J^k(M,K)$, and the natural projection $\pi^k\colon J^k(M,K) \to J^0(M,K) = M \times K$ defines the smooth vector bundle over $M \times K$, which is called k-jet bundle. For every function $u\colon M \to K$ its k-th prolongation $j^k(u)\colon M \to J^k(M,K)$ is naturally defined. k-th order PDE for functions $M \to K$ can be considered as subset E of $J^k(M,K)$; solutions of E are such functions $u\colon M \to K$ that the image of $j^k(u)$ contained in E.

In more general situation we have a smooth fiber bundle $\pi\colon N \to M$ instead of a projection $M \times K \to M$, and sections $s\colon M \to N$ instead of functions $u\colon M \to K$. By $\Gamma\pi$ denote the space of smooth sections of π; remember that a section of π is a map $s\colon M \to N$ such that $\pi \circ s$ is the identity. Definitions of k-jet bundle $\pi^k\colon J^k(\pi) \to J^0(\pi) = N$ and of the k-th prolongation $j^k\colon \Gamma\pi \to \Gamma\left(\pi \circ \pi^k\colon J^k(\pi) \to M\right)$ are the same as ones for the

functions. Let E be a subset of $J^k(\pi)$; it could be considered as k-th order partial differential equation for a sections of π, that is $s \in \Gamma\pi$ is the solution of E if the image of $j^k(s)$ is contained in E.

Suppose $\pi\colon N \to M$, $\pi'\colon N' \to M'$ are smooth fiber bundles, $F\colon \pi \to \pi'$ is a bundle morphism which is diffeomorphism on the fibers (see Fig. 1). F induces the map $F^*\colon \Gamma\pi' \to \Gamma\pi$; denote by $\Gamma_F\pi$ its image. We say that a section of π is **F-projected** if it is contained in $\Gamma_F\pi$. If F is surjective then F^* is injective, so we have the map $F_\#\colon \Gamma_F\pi \to \Gamma\pi'$. If F is surjective submersion then we have the map $F^k\colon J^k_F(\pi) \to J^k(\pi')$, where $J^k_F(\pi) = F^* J^k(\pi')$ is the bundle of k-jets of F-projected sections of π (see Fig. 2). Recall that a map F is called a submersion if $\mathrm{d}F\colon T_xN \to T_{Fx}N'$ is surjective at each point $x \in N$.

Definition 1. *Let $\pi\colon N \to M$, $\pi'\colon N' \to M'$ be a smooth fiber bundles, E be a subset of $J^k(\pi)$, $F\colon \pi \to \pi'$ be a smooth bundle morphism. We say that F **is admitted by** E if the following conditions are satisfied:*

1. *F is a surjective submersion,*
2. *the diagram Fig. 1 is the pullback square in the category of smooth manifolds, that is for any $x \in M$ the map $\pi^{-1}(x) \to \pi'^{-1}(Fx)$ is diffeomorphism,*
3. *$E \cap J^k_F(\pi) = \left(F^k\right)^{-1}(E')$ fore some subset E' of $J^k(\pi')$.*

*We say that E' is F**-projection** of E.*

It turns out that the language of category theory is very convenient for our study of PDE. Recall that a category $\mathcal{C}$ consists of a collection of *objects* $\mathrm{Ob}_{\mathcal{C}}$, a collection of *morphisms* (or *arrows*) $\mathrm{Hom}_{\mathcal{C}}$ and four operations. The first two operations associate with each morphism F of $\mathcal{C}$ its *source* and its *target*, both of which are objects of $\mathcal{C}$. The remaining two operations are an operation that associates with each object $\mathbf{C}$ of $\mathcal{C}$ an *identity morphism* $\mathrm{id}_{\mathbf{C}} \in \mathrm{Hom}_{\mathcal{C}}$ and an operation of *composition* that associates to any pair (F, G) of morphisms of $\mathcal{C}$ such that the source of F is coincide with the target of G another morphism $F \circ G$, their composite. These operations have to satisfy some natural axioms [Mac Lane, 1998].

Definition 2. *$\mathcal{PDE}_0$ is the category whose objects are pairs $\left(\pi\colon N \to M, E \subset J^k(\pi)\right)$, π is a smooth viber bundle, $k \in \mathbb{N}$, and morphisms from the object $\left(\pi\colon N \to M, E \subset J^k(\pi)\right)$ to the object $\left(\pi'\colon N' \to M', E' \subset J^k(\pi')\right)$ are smooth bundle morphisms $F\colon \pi \to \pi'$ admitted by E such that E' is F-projection of E.*

If $F\colon (\pi, E) \to (\pi', E')$ is morphism in $\mathcal{PDE}_0$ then F^* define the bijection between the set of all solutions of E' and the set of all F-projected solutions of E.

3 The category $\mathcal{PDE}$ of partial differential equations

Now we want to define extended category $\mathcal{PDE}$, whose objects are pairs (N, E) where N is a smooth manifold and E is a subset of the bundle $J^k_m(N)$ of k-jets of m-dimensional submanifolds of N.

Let N be a C^r-smooth manifold, $0 < m < \dim N$. The jet bundle $\pi^k\colon J^k_m(N) \to N$ is the fiber bundle with the fiber $J^k_m(N)\big|_x$ over the point $x \in N$, where $J^k_m(N)\big|_x$ is the set of equivalence classes of smooth m-dimensional submanifolds L of N passing through x under the equivalence relation of k-th order contact in x.

k-jet of k-smooth m-dimensional submanifold L over the point $x \in L$ is the equivalence class from $J^k_m(N)\big|_x$ determined by L. Thus we have the prolongation map $j^k\colon L \to J^k_m(N)$ taking each point $x \in L$ to the k-jet of L over x (so it is the section of the fiber bundle $J^k_m(N)$ restricted to $L \subset N$). For every $k > l \geq 0$ there exist natural projection $\pi^{k,l}\colon J^k_m(N) \to J^l_m(N)$ mapping k-jet of L to l-jet of L over x for every m-dimensional submanifold L of N and every $x \in L$.

For a submanifold L of N the differential of the prolongation map $j^k\colon L \to J^k_m(N)$ takes the tangent bundle TL to the tangent bundle $TJ^k_m(N)$. The closure of the union of the images of TL in $TJ^k_m(N)$ when L runs over all m-dimensional submanifolds of N is the vector subbundle of $TJ^k_m(N)$; it is called Cartan distribution on $J^k_m(N)$.

Let E be a submanifold of $J^k(\pi)$, $\pi\colon N \to M$, $m = \dim N$. The graph of a section is m-dimensional submanifold of N so $J^k(\pi)$ is open subspace of $J^k_m(N)$ and E could be considered as the submanifold of $J^k_m(N)$. The extended version of E is defined as the closure of E in $J^k_m(N)$ [Olver, 1993]. Because we don't plan to consider infinitesimal properties of E unlike the Lie group analysis of PDE, we could consider any subsets E of $J^k_m(N)$ as a partial differential equations. By definition, **solutions** of such an equation are smooth m-dimensional non-vertical integral manifolds of the Cartan distribution on $J^k_m(N)$ contained in E. Note that for any m-dimensional submanifold L of N its prolongation $j^k(L)$ is non-vertical integral manifold of the Cartan distribution on $J^k_m(N)$. So if $E \subset J^k_m(N)$ is obtained from the traditional PDE as it was described above, and L is the graph of a section u of π, then L is the solution of E in the above sense if and only if u is the solution of corresponding traditional PDE in the traditional sense. In addition there is allowed the possibility of both multiply-valued solutions and solutions with infinite derivatives (see [Olver, 1993] for the details). Wherever we write concrete equations in the traditional form below we mean the extended versions of these equations that is closure of the corresponding set in $J^k_m(N)$.

Now let us introduce some auxiliary notations.

Let $F\colon N \to N'$ be a map. We will say that $L \subset N$ is F-**projected** if $L = F^{-1}(F(L))$. Note that if F is a surjective submersion and L is F-projected submanifold of N then $L' = F(L)$ is the submanifold of N'.

Let N, N' be C^r-smooth manifolds, $0 < m < \dim N$. Let $F\colon N \to N'$ be a surjective submersion of smoothness class C^s, $k \leq s \leq r$.

Definition 3. *F**-projected jet bundle** $J^k_{m,F}(N)$ is the submanifold of $J^k_m(N)$ that consists of k-jets of all m-dimensional F-projected submanifolds of N.*

We will write $J^k_F(N)$ instead of $J^k_{m,F}(N)$ if the value of m is clear from the context.

There is natural isomorphism between the bundles $J^k_{m,F}(N)$ and $F^*J^k_{m'}(N')$ over N, where $F^*J^k_{m'}(N') = J^k_{m'}(N') \times_{N'} N$ is the pullback of $J^k_{m'}(N')$ by F, $\dim N - m = \dim N' - m'$. Therefore we can lift the map F to the map $F^k\colon J^k_{m,F}(N) \to J^k_{m'}(N')$ by the following natural way. Suppose $\vartheta \in J^k_{m,F}(N)$.

1. Take an arbitrary F-projected submanifold L of N such that the k-prolongation of L pass through ϑ (that is k-jet of L over the point $\pi^k(\vartheta)$ is ϑ).

2. Assign to ϑ the point $\vartheta' \in J^k_{m'}(N')$, where ϑ' is k-jet of the submanifold $L' = F(L)$ of N' over the point $F \circ \pi^k(\vartheta)$.

Definition 4. *Let $E \subset J^k_m(N)$. Let $F\colon N \to N'$ be a smooth surjective submersion. We say that F **is admitted by** E if the intersection $E \cap J^k_{m,F}(N)$ is F^k-projected subset of $J^k_{m,F}(N)$ (see Fig. 3(a)). Equivalently, $E \cap J^k_{m,F}(N)$ is the pre-image $\left(F^k\right)^{-1}(E')$ of some $E' \subset J^k_{m'}(N')$; we say that E' is F**-projection** of E.*

$$
\begin{tikzcd}
E \arrow[d, tail] & E \cap J^k_{m,F}(N) \arrow[l, tail] \arrow[r, two heads] \arrow[d] & E' \arrow[d, tail] \\
J^k_m(N) \arrow[d, two heads, "\pi^k"'] & J^k_{m,F}(N) \arrow[l, tail] \arrow[r, two heads, "F^k"] \arrow[d, two heads] & J^k_{m'}(N') \arrow[d, two heads, "\pi'^k"'] \\
N \arrow[r, equal] & N \arrow[r, two heads, "F"'] & N'
\end{tikzcd}
$$

a

$$
\begin{tikzcd}
E \arrow[d, tail] & E \cap \Delta \arrow[l, tail] \arrow[r, two heads] \arrow[d, tail] & E' \arrow[d, tail] \\
J^k_m(N) \arrow[d, two heads, "\pi^k"'] & \Delta \arrow[l, tail] \arrow[r, two heads, "\tilde{F}"] \arrow[d, two heads] & J^{k'}_{m'}(N') \arrow[d, two heads, "\pi'^{k'}"'] \\
N \arrow[r, equal] & N & N'
\end{tikzcd}
$$

b

Figure 3: Morphisms of $\mathcal{PDE}$ (a) and morphisms of $\mathcal{PDE}_{\mathrm{ext}}$ (b)

Definition 5. *A **category of partial differential equations** $\mathcal{PDE}$ is defined as follows:*

- *objects of $\mathcal{PDE}$ are pairs (N, E), where N is a smooth manifold, E is a subset of $J^k_m(N)$ for some integer $k, m \geq 1$;*
- *morphisms of $\mathcal{PDE}$ with source $\mathbf{A} = (N, E)$ are all surjective submersions $F\colon N \to N'$ admitted by E; target of this morphism is (N', E') where E' is F-projection of E;*
- *the identity morphism from $\mathbf{A}$ is the identity mapping of N, composition of morphisms is composition of appropriate maps.*

If $F\colon (N, E) \to (N', E')$ is morphism in $\mathcal{PDE}$ then F^* define the bijection between the set of all solutions of E' and the set of all F^k-projected solutions of E.

In [Prokhorova, 1998] we defined the following notion of a map admitted by a pair of equations: a map $F\colon N \to N'$ is admitted by an ordered pair of equations $(\mathbf{A}, \mathbf{A}')$, $\mathbf{A} = (N, E)$, $\mathbf{A}' = (N', E')$ if for any $L' \subset N'$ the following two conditions are equivalent:

- L' is the graph of a solution of E',
- $F^{-1}(L')$ is the graph of a solution of E.

However, we are not happy with this definition; in particular, because it deals only with global solutions of E. Therefore we now formulate the notion of a map admitted by a equation in terms of (locally defined) jet bundles.

Remark 1. *Let $\mathbf{A} = (N, E)$ be an object of $\mathcal{PDE}$. Then its automorphism group* $\mathrm{Aut}(\mathbf{A})$ *is the symmetry group for the equation E.*

Remark 2. *Suppose G is a subgroup of the symmetry group of E such that N/G is a smooth manifold. Then the natural projection $N \to N/G$ define the morphism $(N, E) \to (N/G, E/G)$ in $\mathcal{PDE}$ where E/G is the equation describing G-invariant solutions of E.*

Therefore, reduction of E by subgroups of $\mathrm{Aut}(\mathbf{A})$ defines the part of nontrivial morphisms from $\mathbf{A}$. But the class of all morphisms from $\mathbf{A}$ is significantly richer then the class of morphisms arising from reduction by subgroups of $\mathrm{Aut}(\mathbf{A})$. Let $\mathrm{Sol}(\mathbf{A})$ be the set of all solutions of $\mathbf{A}$, that is of all smooth m-dimensional non-vertical integral manifolds of the Cartan distribution on $J^k_m(N)$ contained in E. In general, the subset $F^*(\mathrm{Sol}(\mathbf{A}')) = \{F^{-1}(L')\colon L' \in \mathrm{Sol}(\mathbf{A}')\} \subseteq \mathrm{Sol}(\mathbf{A})$ of solutions of $\mathbf{A}$ arising from a morphism $F\colon \mathbf{A} \to \mathbf{A}'$ can *not* be represented as a set of solutions that are invariant under some subgroup of $\mathrm{Aut}(\mathbf{A})$. In particular, $F^*(\mathrm{Sol}(\mathbf{A}'))$ can be the set of G-invariant solutions, where G is a transformation group that is not necessarily a symmetry group of E. Moreover, for a morphism $F\colon \mathbf{A} \to \mathbf{A}'$ it may occur that for every nontrivial diffeomorphism g of N there exist an element in $F^*(\mathrm{Sol}(\mathbf{A}'))$ that is not g-invariant. More detailed discussion is given in Section 9; see also [Prokhorova, 2000], [Prokhorova, 2001].

Our approach is conceptually close to the approach developed in [Elkin, 1999] that deals with control systems. If we set aside the control part and look at this approach relative to ordinary differential equations, then we get the category of ordinary differential equations, whose objects are ODE systems of the form $\dot{x} = \xi$, $x \in X$, where X is a manifold equipped with a vector field ξ, and morphism from a system $\mathbf{A}$ to a system $\mathbf{A}'$ is a smooth map F from the phase space X of $\mathbf{A}$ to the phase space X' of $\mathbf{A}'$ that projected ξ to ξ'. In other words, F is a morphism if it transforms solutions (phase trajectories) of $\mathbf{A}$ to the solutions of $\mathbf{A}'$: $F_*(\mathrm{Sol}(\mathbf{A})) = \mathrm{Sol}(\mathbf{A}')$.

By contrast, we deal with pullbacks of the solutions of the quotient equation $\mathbf{A}'$ to the solutions of the original equation $\mathbf{A}$. In our approach the number of dependent variables in the reduced PDE remains the same, while the number of independent0 variables is not increased. Thus in the approach proposed the quotient object notion is an analogue of the sub-object notion (in terminology of [Elkin, 1999]) with respect to the information about the solutions of the given equation; however, it is similar to the quotient object notion with respect to interrelations between the given and reduced equations.

Note also that described above category of ODE from [Elkin, 1999] is isomorphic to certain subcategory of $\mathcal{PDE}$. Namely, let us consider the following subcategory $\mathcal{PDE}_1$ of $\mathcal{PDE}$:

- objects of $\mathcal{PDE}_1$ are pairs (N, E), where $N = X \times \mathbb{R}$, E is a first order linear PDE of the form $L_\xi u = 1$ for unknown function $u\colon X \to \mathbb{R}$, $\xi \in TX$;
- morphisms of $\mathcal{PDE}_1$ are morphisms of $\mathcal{PDE}$ of the form $(x, u) \mapsto (x'(x), u)$.

One can easy see that the category of ODE from [Elkin, 1999] is isomorphic to $\mathcal{PDE}_1$: the object $L_\xi u = 1$ corresponds to the object $\dot{x} = \xi$, and the morphism $(x, u) \mapsto (x'(x), u)$ corresponds to the morphism $x \mapsto x'(x)$.

The category of differential equations was also defined in [Krasil'shchik, 1999] in a different way: objects are infinite-dimensional manifolds endowed with integrable finite-dimensional distribution (particularly, infinite prolongations of differential equations), and

morphisms are smooth maps such that image of the distribution is contained in the distribution on the image, similarly to morphisms in [Elkin, 1999]. Thus, the category of differential equations defined in [Krasil'shchik, 1999] is quite different from the category $\mathcal{PDE}$ defined here; one should keep it in mind in order to avoid confusion. The factorization of PDE $\mathbf{A}$ by a symmetry group described in [Krasil'shchik, 1999] is PDE $\mathbf{A}'$ on the quotient space that described images of all solutions of $\mathbf{A}$ at the projection to the quotient space: $F_*(\mathrm{Sol}(\mathbf{A})) = \mathrm{Sol}(\mathbf{A}')$. In that approach every factorization of $\mathbf{A}$ provides a part of the information about all the solutions of $\mathbf{A}$. In our approach factorization of $\mathbf{A}$ is such an equation $\mathbf{A}'$ that the pullbacks of its solutions are solutions of $\mathbf{A}$: $F^*(\mathrm{Sol}(\mathbf{A}')) \subseteq \mathrm{Sol}(\mathbf{A})$; so that from every factorization we obtain the full information about a certain set of the solutions of the given equation.

Lemma 1. *All morphisms in $\mathcal{PDE}$ are epimorphisms.*

Lemma 2. *Suppose (N, E), (N', E'), (N'', E'') are objects of $\mathcal{PDE}$, $F\colon N \to N'$ is a morphism from (N, E) to (N', E') in $\mathcal{PDE}$, $G\colon N' \to N''$ is surjective submersion. Then the following two conditions are equivalent (see Fig. 4):*

- *G is a morphism from (N', E') to (N'', E''),*
- *GF is a morphism from (N, E) to (N'', E'').*

$$
\begin{tikzcd}
E \cap J^k_{GF}(N) \arrow[d, tail] \arrow[dr, two heads] \arrow[ddrr, bend left, "(GF)^k"] & & \\
J^k_{GF}(N) \arrow[d, tail] \arrow[dr, two heads] & E' \cap J^k_G(N') \arrow[d, tail] \arrow[dr, two heads] & \\
J^k_F(N) \arrow[d, tail] \arrow[dr, two heads, "F^k"] & J^k_G(N') \arrow[d, tail] \arrow[dr, two heads, "G^k"] & E'' \arrow[d, tail] \\
J^k(N) \arrow[d, two heads, "\pi_k"'] & J^k(N') \arrow[d, two heads, "\pi'_k"'] & J^k(N'') \arrow[d, two heads, "\pi''_k"'] \\
N \arrow[r, two heads, "F"'] & N' \arrow[r, two heads, "G"'] & N''
\end{tikzcd}
$$

Figure 4: Diagram for Lemma 2

4 The extended category $\mathcal{PDE}_{\mathrm{ext}}$ of partial differential equations

Note that the Cartan distribution $C^k(N)$ on $J^k_m(N)$ restricted to $J^k_{m,F}(N)$ coincides with the lifting $\left(F^k\right)^* C^k_{m'}(N')$ of the Cartan distribution on $J^k_{m'}(N')$, $m' = m - \dim N + \dim N'$.

Taking this into account and using the analogy with higher symmetry group, we replace $J^k_{m,F}(N)$ to arbitrary submanifold Δ of $J^k_m(N)$. Thus we obtain the category $\mathcal{PDE}_{\text{ext}}$ with the same objects as $\mathcal{PDE}$ and extended set of morphisms involving transformations of jets. (This category will not be used in the rest of the paper.)

Definition 6. *An* ***extended category of partial differential equations*** $\mathcal{PDE}_{\text{ext}}$ *is defined as follows:*

- *objects of* $\mathcal{PDE}_{\text{ext}}$ *are pairs* (N, E)*, where* N *is a smooth manifold,* E *is a subset of* $J^k_m(N)$ *for some integer* $k, m \geq 1$*;*
- *morphisms of* $\mathcal{PDE}_{\text{ext}}$ *from* $\mathbf{A} = (N, E \subset J^k_m(N))$ *to* $\mathbf{A}' = (N', E' \subset J^{k'}_{m'}(N'))$ *are all pairs* $\left(\Delta, \tilde{F}\right)$ *such that* Δ *is a smooth submanifold of* $J^k_m(N)$*,* $\tilde{F}\colon \Delta \to J^{k'}_{m'}(N')$ *is a surjective submersion, the Cartan distribution on* $J^k_m(N)$ *restricted to* Δ *coincide with the lifting* $\tilde{F}^* C^{k'}_{m'}(N')$ *of the Cartan distribution on* $J^{k'}_{m'}(N')$*, and* $E \cap \Delta = \tilde{F}^{-1}(E')$ *(see Fig. 3(b))*
- *the identity morphism from* $\mathbf{A}$ *is* $\left(\Delta = J^k_m(N), \tilde{F} = \mathrm{id}_N\right)$*, composition of* $\left(\Delta \subset J^k_m(N), \tilde{F}\colon \Delta \to J^{k'}_{m'}(N')\right)$ *and* $\left(\Delta' \subset J'^{k'}_m(N'), \tilde{F}'\colon \Delta' \to J^{k''}_{m''}(N'')\right)$ *is* $\left(\tilde{F}^{-1}(\Delta'), \tilde{F}' \circ \tilde{F}\right)$*.*

For each integral manifold of the Cartan distribution on E' its preimage is an integral manifold of the Cartan distribution on E, so for each solution of E' its pullback is some solution of E.

$\mathcal{PDE}$ embeds to $\mathcal{PDE}_{\text{ext}}$ by the following natural way: to the morphisms $F\colon N \to N'$ of $\mathcal{PDE}$ from the equation of k-th order we assign the morphisms $\left(\Delta, \tilde{F}\right)$ of $\mathcal{PDE}_{\text{ext}}$ such that $\Delta = J^k_{m,F}(N)$, $\tilde{F} = F^k$.

5 Usage of subcategories

We start with review of some basic definitions of category theory [Mac Lane, 1998]. Given a category $\mathcal{C}$ and an object $\mathbf{A}$ of $\mathcal{C}$, one may construct the category $(\mathbf{A} \downarrow \mathcal{C})$ of objects under $\mathbf{A}$ (this is the particular case of the comma category): objects of $(\mathbf{A} \downarrow \mathcal{C})$ are morphisms of $\mathcal{C}$ with source $\mathbf{A}$, and morphisms of $(\mathbf{A} \downarrow \mathcal{C})$ from one such object $F\colon \mathbf{A} \to \mathbf{B}$ to another $F'\colon \mathbf{A} \to \mathbf{B}'$ are morphisms $G\colon \mathbf{B} \to \mathbf{B}'$ of $\mathcal{C}$ such that $F' = G \circ F$.

Suppose $\mathcal{C}$ is a subcategory of $\mathcal{PDE}$, $\mathbf{A}$ is an object of $\mathcal{C}$. Then the category $(\mathbf{A} \downarrow \mathcal{C})$ of objects under $\mathbf{A}$ describes collection of quotient equations for $\mathbf{A}$ and their interconnection in the framework of $\mathcal{C}$.

To each morphism $F\colon \mathbf{A} \to \mathbf{B}$ with source $\mathbf{A}$ (that is to each object of the comma category $(\mathbf{A} \downarrow \mathcal{C})$) assign the set $F^*(\mathrm{Sol}(\mathbf{B})) \subseteq \mathrm{Sol}(\mathbf{A})$ of such solutions of $\mathbf{A}$ that "projected" onto underying space of $\mathbf{B}$ (space of dependent and independent variables). We can identify such morphisms that generated the same sets of solutions of $\mathbf{A}$, that is identify isomorphic objects of the comma category $(\mathbf{A} \downarrow \mathcal{C})$.

Describe the situation more explicitly. An equivalence class of epimorphisms with source $\mathbf{A}$ is called a quotient object of $\mathbf{A}$, where two epimorphisms $F\colon \mathbf{A} \to \mathbf{B}$

and $F' \colon \mathbf{A} \to \mathbf{B}'$ are equivalent if $F' = I \circ F$ for some isomorphism $I \colon \mathbf{B} \to \mathbf{B}'$ [Mac Lane, 1998]. If $F \colon \mathbf{A} \to \mathbf{B}$ and $F' \colon \mathbf{A} \to \mathbf{B}'$ are equivalent, then they lead to the same subsets of the solutions of $\mathbf{A}$: $F^*(\mathrm{Sol}(\mathbf{B})) = F'^*(\mathrm{Sol}(\mathbf{B}'))$. So if we interested only in the sets of the solutions of $\mathbf{A}$, then all representatives of the same quotient object have the same rights.

Therefore, the following problems naturally arise:

- to study all morphisms with given source,
- to choose a "simplest" representative from every equivalence class, or to choose representative with the simplest target (that is the simplest quotient equation).

In order to deal with these problems, we develop a special-purpose language.

Let us introduce a number of partial orders on the class of all categories to describe arising situations. First of all, we define a few types of subcategories.

Definition 7. *Suppose $\mathcal{C}$ is a category, $\mathcal{C}_1$ is a subcategory of $\mathcal{C}$.*

- *$\mathcal{C}_1$ is called a **wide** subcategory of $\mathcal{C}$ if all objects of $\mathcal{C}$ are objects of $\mathcal{C}_1$.*
- *$\mathcal{C}_1$ is called a **full** subcategory of $\mathcal{C}$ if every morphism in $\mathcal{C}$ with source and target from $\mathcal{C}_1$ is a morphism in $\mathcal{C}_1$.*
- *We say that $\mathcal{C}_1$ is **full under isomorphisms in** $\mathcal{C}$ if every isomorphism in $\mathcal{C}$ with source and target from $\mathcal{C}_1$ is an isomorphism in $\mathcal{C}_1$.*
- *We say that $\mathcal{C}_1$ is **closed in** $\mathcal{C}$ if every morphism in $\mathcal{C}$ with source from $\mathcal{C}_1$ is a morphism in $\mathcal{C}_1$. (Note that every subcategory that is closed in $\mathcal{C}$ is full in $\mathcal{C}$.)*
- *We say that $\mathcal{C}_1$ is **closed under isomorphisms in** $\mathcal{C}$ if every isomorphism in $\mathcal{C}$ with source from $\mathcal{C}_1$ is an isomorphism in $\mathcal{C}_1$.*
- *We say that $\mathcal{C}_1$ is **dense in** $\mathcal{C}$ if every object of $\mathcal{C}$ is isomorphic in $\mathcal{C}$ to an object of $\mathcal{C}_1$.*
- *We say that $\mathcal{C}_1$ is **plentiful in** $\mathcal{C}$ if for every morphism $F \colon \mathbf{A} \to \mathbf{B}$ in $\mathcal{C}$, $\mathbf{A} \in \mathrm{Ob}_{\mathcal{C}_1}$, there exist an isomorphism $I \colon \mathbf{B} \to \mathbf{C}$ in $\mathcal{C}$ such that $I \circ F \in \mathrm{Hom}_{\mathcal{C}_1}$ (in other words, for every quotient object of $\mathbf{A}$ in $\mathcal{C}$ there exist a representative of this quotient object in $\mathcal{C}_1$). Such morphism $I \circ F$ we call $\mathcal{C}_1$-**canonical** for F.*
- *We say that $\mathcal{C}_1$ is **fully dense (fully plentiful) in** $\mathcal{C}$ if $\mathcal{C}_1$ is a full subcategory of $\mathcal{C}$ and $\mathcal{C}_1$ is dense (plentiful) in $\mathcal{C}$.*

The first two parts of the Definition are standard notions of category theory, but the notions of the other parts are introduced here for the sake of description of the $\mathcal{PDE}$ structure.

Remark 3. *Using the notion of "the category of objects under $\mathbf{A}$", we can define the notions of closed subcategory and plentiful subcategory by the following way:*

- *$\mathcal{C}_1$ is closed in $\mathcal{C}$ if for each $\mathbf{A} \in \mathrm{Ob}_{\mathcal{C}_1}$ the category $(\mathbf{A} \downarrow \mathcal{C}_1)$ is wide in $(\mathbf{A} \downarrow \mathcal{C})$.*
- *$\mathcal{C}_1$ is plentiful in $\mathcal{C}$ if for each $\mathbf{A} \in \mathrm{Ob}_{\mathcal{C}_1}$ the category $(\mathbf{A} \downarrow \mathcal{C}_1)$ is dense in $(\mathbf{A} \downarrow \mathcal{C})$.*

Remark 4. *$\mathcal{C}_1$ is fully dense in $\mathcal{C}$ if and only if the embedding functor $\mathcal{C}_1 \to \mathcal{C}$ defines an equivalence of these categories.*

Choose some category $\mathcal{U}$, which is big enough to contain all needful for us categories as it's subcategories. For our purposes $\mathcal{U} = \mathcal{PDE}$ will be sufficient.

Define the category $\mathcal{U}_{\geq}$ whose objects are subcategories $\mathcal{C}$ of $\mathcal{U}$, and a collection $\mathrm{Hom}_{\mathcal{U}_{\geq}}(\mathcal{C}_1, \mathcal{C}_2)$ of morphisms from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a one-element set if $\mathcal{C}_2$ is subcategory of $\mathcal{C}_1$ and empty otherwise, so an arrow from $\mathcal{C}_1$ to $\mathcal{C}_2$ in $\mathcal{U}_{\geq}$ means that $\mathcal{C}_2$ is the subcategory of $\mathcal{C}_1$. Let $\mathcal{U}_{=}$ be the discrete wide subcategory of $\mathcal{U}_{\geq}$, that is objects of $\mathcal{U}_{=}$ are all subcategories $\mathcal{C}$ of $\mathcal{U}$, and the only morphisms are identities, so $\mathcal{C}_1$ and $\mathcal{C}_2$ are connected by arrow in $\mathcal{U}_{=}$ only if $\mathcal{C}_1 = \mathcal{C}_2$.

Definition 8. *Suppose $\mathcal{C}_1$, $\mathcal{C}_2$ are subcategories of $\mathcal{U}$. A subcategory of $\mathcal{U}$, whose objects are objects of $\mathcal{C}_1$ and $\mathcal{C}_2$ simultaneously, and whose morphisms are morphisms of $\mathcal{C}_1$ and $\mathcal{C}_2$ simultaneously, is called an **intersection** of $\mathcal{C}_1$ and $\mathcal{C}_2$ and is denoted by $\mathcal{C}_1 \cap \mathcal{C}_2$.*

In other words, $\mathcal{C}_1 \cap \mathcal{C}_2$ is the fibered sum of $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{U}_{\geq}$.

Lemma 3. *Suppose $\mathcal{C}_1$ is closed in $\mathcal{C}$, and $\mathcal{C}_2$ is (full/closed/dense/plentiful) subcategory of $\mathcal{C}$; then $\mathcal{C}_1 \cap \mathcal{C}_2$ is closed in $\mathcal{C}_2$ and is (full/closed/dense/plentiful) subcategory of $\mathcal{C}_1$.*

Now we introduce some graphic designations for various types of subcategories of $\mathcal{U}_{\geq}$ (see Table 1). These designations will be used, particularly, for the representation of the structure of the category of parabolic equations described below.

	$\mathfrak{W}$	Wide
	$\mathfrak{F}$	Full
	$\mathfrak{F}_{\mathrm{I}}$	Full under isomorphisms
	$\mathfrak{C}$	Close
	$\mathfrak{C}_{\mathrm{I}}$	Close under isomorphisms
	$\mathfrak{D}$	Dense
	$\mathfrak{P}$	Plentiful

Table 1: Basic meta-categories (arrows)

We shall use the term "meta-category" both for the category $\mathcal{U}_{\geq}$ and for its subcategories defined below to avoid confusion between $\mathcal{U}_{\geq}$ and "ordinary" categories which are objects of $\mathcal{U}_{\geq}$; and we shall use Gothic script for meta-categories except $\mathcal{U}_{\geq}$. One may view these meta-categories as a partial orders on the class of all subcategories of $\mathcal{U}$; we prefer category terminology here since this allow us to use category constructions for the interrelations between various partial orders.

Define $\mathfrak{W}$, $\mathfrak{F}$, $\mathfrak{F}_{\mathrm{I}}$, $\mathfrak{C}$, $\mathfrak{C}_{\mathrm{I}}$, $\mathfrak{D}$, and $\mathfrak{P}$ that are wide subcategories of meta-category Ug. Objects of them are categories, but arrows from $\mathcal{C}_1$ to $\mathcal{C}_2$ have a different meaning:

- in the meta-category $\mathfrak{W}$ it mean that $\mathcal{C}_2$ is wide subcategory of $\mathcal{C}_1$,
- in the meta-category $\mathfrak{F}$ it means that $\mathcal{C}_2$ is full subcategory of $\mathcal{C}_1$,
- in the meta-category $\mathfrak{F}_{\mathrm{I}}$ it means that $\mathcal{C}_2$ is full under isomorphisms in $\mathcal{C}_1$,
- in the meta-category $\mathfrak{C}$ it means that $\mathcal{C}_2$ is closed subcategory of $\mathcal{C}_1$,
- in the meta-category $\mathfrak{C}_{\mathrm{I}}$ it means that $\mathcal{C}_2$ is closed under isomorphisms in $\mathcal{C}_1$,

- in the meta-category $\mathfrak{D}$ it means that $\mathcal{C}_2$ is dense subcategory of $\mathcal{C}_1$,
- in the meta-category $\mathfrak{P}$ it means that $\mathcal{C}_2$ is plentiful subcategory of $\mathcal{C}_1$,

We shall denote the intersections of these meta-categories by the concatenations of appropriate letters, for example: $\mathfrak{FD} = \mathfrak{F} \cap \mathfrak{D}$.

Lemma 4.

- $\mathfrak{F}_{\mathrm{I}} \cap \mathfrak{P} = \mathfrak{F} \cap \mathfrak{P}$,
- $\mathfrak{C}_{\mathrm{I}} \cap \mathfrak{P} = \mathfrak{C}$,
- $\mathfrak{F} \cap \mathfrak{P} \cap \mathfrak{D} = \mathfrak{F} \cap \mathfrak{D}$.

Interrelations between "basic" meta-categories $\mathfrak{W}$, $\mathfrak{F}$, $\mathfrak{F}_{\mathrm{I}}$, $\mathfrak{C}$, $\mathfrak{C}_{\mathrm{I}}$, $\mathfrak{D}$, $\mathfrak{P}$ and their intersections ("composed" meta-categories) are represented on Fig. 5(a). Here an arrow means the predicate "to be subcategory of"; we shall call it "the meta-arrow". For example, meta-arrow from $\mathfrak{D}$ to $\mathfrak{W}$ means that $\mathfrak{W}$ is subcategory of $\mathfrak{D}$. In the language of "ordinary" categories this meta-arrow means that the statement "$\mathcal{C}_2$ is wide in $\mathcal{C}_1$" implies that $\mathcal{C}_2$ is dense in $\mathcal{C}_1$. Everywhere on Fig. 5(a) a pair of meta-arrows with the same target means that this meta-category (target of these meta-arrows) is the intersection of two "top" meta-categories (sources of these meta-arrows). For example, $\mathfrak{FD} = \mathfrak{FP} \cap \mathfrak{PD}$.

On Fig. 5(b) the same scheme is represented as on Fig. 5(a), but letter names are replaced by the arrows of various types.

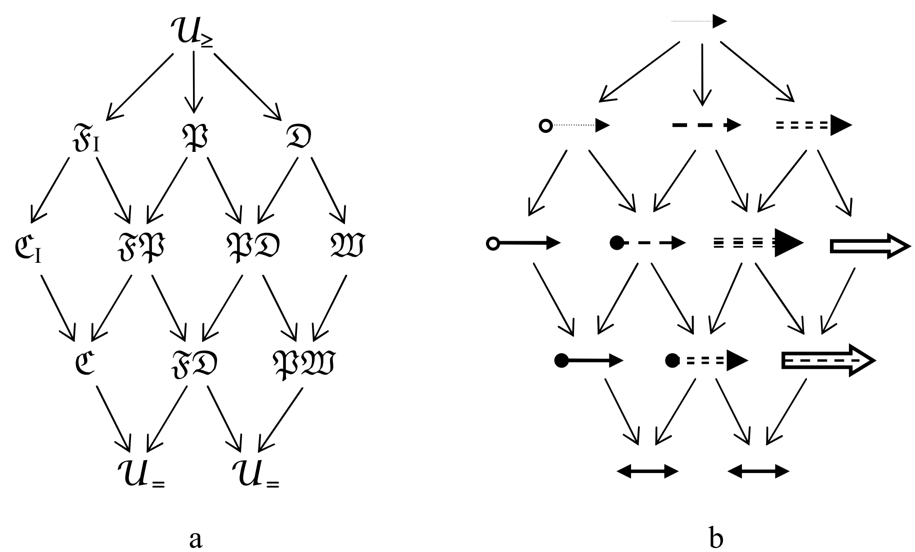

Figure 5: Interrelations between basic meta-categories (arrows) and their intersections

Instead of the investigation of all or the simplest morphisms with the given source, we want to introduce a certain structure in $\mathcal{PDE}$, so that the position of an object in it gives information on the morphisms from the object and on the kind of the simplest representatives of equivalence classes of the morphisms. We construct this structure by means of choosing some subcategories in $\mathcal{PDE}$ and connecting them by the arrows from Fig. 5(b). The part of obtained structure of the category of parabolic equations is shown on Fig. 6. We use this structure also in Section 8 to investigate nonlinear reaction-diffusion equation.

6 The category of parabolic equations

Let us consider the class $P\left(X,T,\Omega\right)$ of differential operators on a connected smooth manifold X, dependent on the time t as on parameter, that are of the form

$$Lu=\sum_{i,j} b^{ij}(t,x,u)u_{ij}+\sum_{i,j} c^{ij}(t,x,u)u_iu_j+\sum_i b^i(t,x,u)u_i+q(t,x,u),$$
$$x\in X,\ t\in T,\ u\in\Omega$$

in some neighborhood of each point, in some (and then arbitrary) local coordinates (x^i) on X. Here subscript i denotes partial derivative with respect to x^i, quadratic form $b^{ij}=b^{ji}$ is positive definite, $c^{ij}=c^{ji}$. Both T and Ω may be bounded, semibounded or unbounded open intervals of $\mathbb{R}$.

Definition 9. ***The category of parabolic equations of the second order*** *$\mathcal{PE}$ is a subcategory of $\mathcal{PDE}$, whose objects are pairs $\mathbf{A}=(N,E)$, $N=T\times X\times\Omega$, where X is a connected smooth manifold, T and Ω are open intervals, E is an equation of the form $u_t=Lu$, $L\in P\left(X,T,\Omega\right)$ (more exactly, E is the extended version of an equation $u_t=Lu$, that is closed submanifold of $J^2_{n+1}(T\times X\times\Omega)$, $n=\dim X$).*

Example 1. *Let $\Phi_k(x)$, $x\in\mathbb{R}^3-\{0\}$ be a spherical harmonic of the k-th order. Then the map $(t,x,u)\mapsto\left(t,|x|\,,u\,/\Phi_k(x)\right)$ defined the morphism in the category $\mathcal{PE}$ from the object $\mathbf{A}$ corresponding to equation $u_t=\Delta u$ and $X=\mathbb{R}^3-\{0\}$, $T=\Omega=\mathbb{R}$, to the object $\mathbf{A}'$ corresponding to equation $u'_{t'}=u'_{x'x'}-k\left(k+1\right)x'^{-2}u'$ and $X'=\mathbb{R}_+$, $T'=\Omega'=\mathbb{R}$. One may assign to the set $\mathrm{Sol}(\mathbf{A}')$ of all solutions of the quotient equation the set $F^*(\mathrm{Sol}(\mathbf{A}'))$ of such solutions of the original equation that may be written in the form $u=\Phi_k(x)u'\left(t,|x|\right)$.*

Example 2. *The following example shows that not every endomorphism in $\mathcal{PE}$ is an automorphism. Consider object $\mathbf{A}$, for which $X=S^1=\mathbb{R}\bmod 1$, $T=\Omega=\mathbb{R}$, $E\colon u_t=u_{xx}$. Then morphism from $\mathbf{A}$ to $\mathbf{A}$ defined by the map $(t,x,u)\mapsto(4t,2x,u)$ has no inverse.*

Theorem 1. *Every morphism in $\mathcal{PE}$ is of the form*

$$(t,x,u)\mapsto\left(t'(t),x'\left(t,x\right),u'(t,x,u)\right),\tag{1}$$

with submersive $t'(t)$, $x'(t,x)$, $u'\left(t,x,u\right)$. Isomorphisms in the category $\mathcal{PE}$ are exactly diffeomorphisms of the form (1).

7 Certain subcategories of $\mathcal{PE}$: classification of the parabolic equations

Certain parts of the $\mathcal{PE}$ structure described below are depicted schematically on Fig. 6. The full structure is not depicted here in view of its awkwardness.

Let us consider five full subcategories $\mathcal{PE}_k$ of the category $\mathcal{PE}$, $1\le k\le 5$, whose

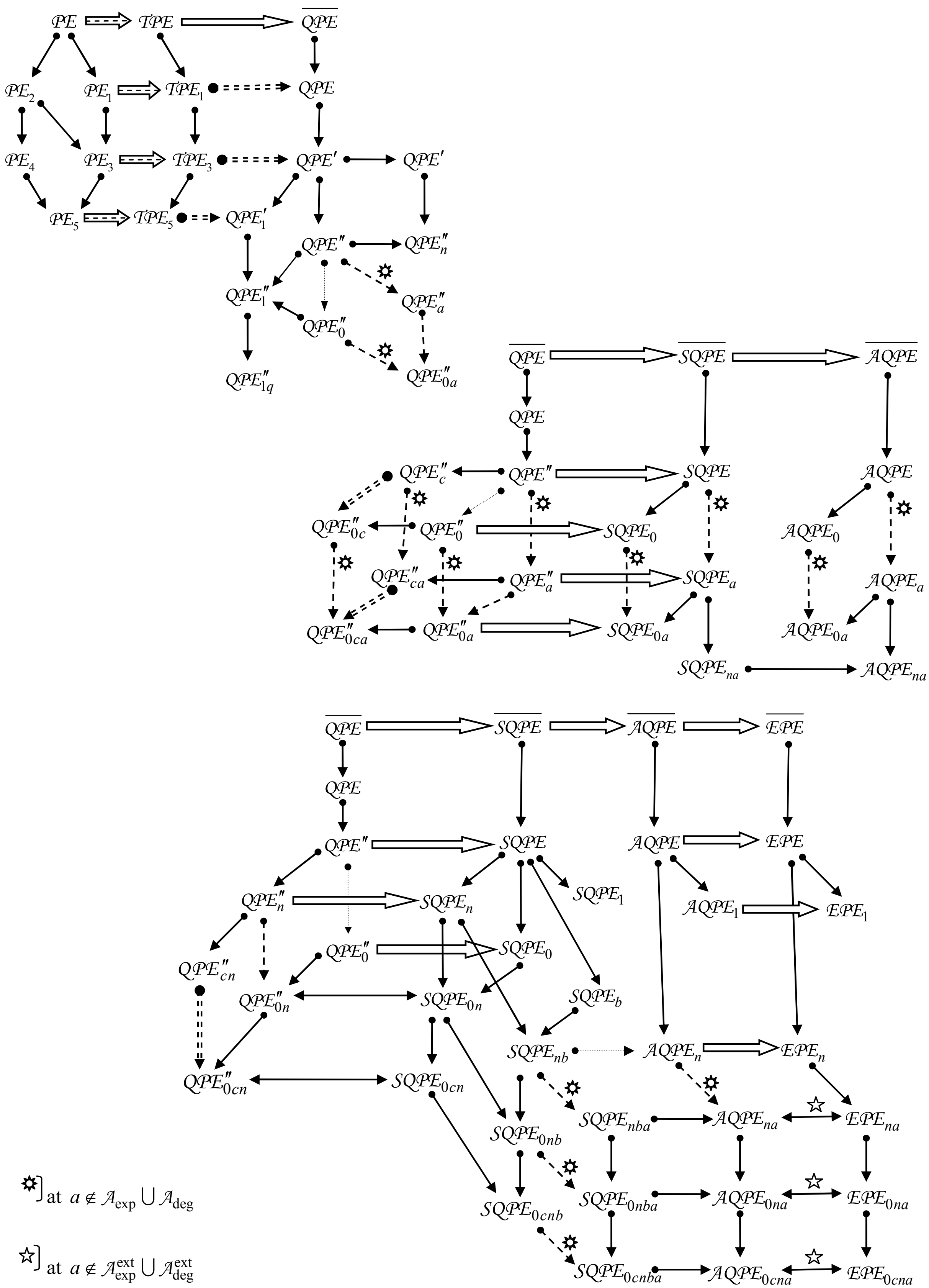

Figure 6: The part of the structure of the category of parabolic equations

objects are equations that can be written locally in the following form:

$$u_t = \sum_{i,j} b^{ij}(t,x,u)\left(u_{ij} + \lambda(t,x,u)u_iu_j\right) + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{PE}_1$}$$

$$u_t = a(t,x,u)\sum_{i,j} \bar{b}^{ij}(t,x)u_{ij} + \sum_{i,j} c^{ij}(t,x,u)u_iu_j + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{PE}_2$}$$

$$u_t = a(t,x,u)\sum_{i,j} \bar{b}^{ij}(t,x)\left(u_{ij} + \lambda(t,x,u)u_iu_j\right) + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{PE}_3$}$$

$$u_t = \sum_{i,j} b^{ij}(t,x)u_{ij} + \sum_{i,j} c^{ij}(t,x,u)u_iu_j + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{PE}_4$}$$

$$u_t = \sum_{i,j} b^{ij}(t,x)\left(u_{ij} + \lambda(t,x,u)u_iu_j\right) + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{PE}_5$}$$

Remark 5. *Everywhere in the paper we use notation of a category equipped with a subscript and/or primes for its full subcategory. For example, defined below $\mathcal{QPE}_k$, $\mathcal{QPE}'$ and $\mathcal{QPE}'_k$ are full subcategories of $\mathcal{QPE}$.*

Remark 6. *Note that in equations of the categories $\mathcal{PE}_2$ and $\mathcal{PE}_3$ a function a is determined up to multiplication by arbitrary function from $T \times X$ to $\mathbb{R}^+$; moreover, it is determined only locally. Nevertheless we can lead these equations to the equations of the same form but with globally defined function $a\colon T \times X \times \Omega \to \mathbb{R}^+$. For example, we can require that conditions $a(t,x,u_0) \equiv 1$ take place, where u_0 is arbitrary point of Ω. Everywhere below we shall assume that function a is globally determined on $T \times X \times \Omega$.*

Theorem 2.

1. *$\mathcal{PE}_1$ and $\mathcal{PE}_2$ are closed in $\mathcal{PE}$.*
2. *$\mathcal{PE}_3 = \mathcal{PE}_1 \cap \mathcal{PE}_2$ is closed in $\mathcal{PE}_1$, in $\mathcal{PE}_2$, and in $\mathcal{PE}$.*
3. *$\mathcal{PE}_4$ is closed in $\mathcal{PE}_2$ and in $\mathcal{PE}$.*
4. *$\mathcal{PE}_5 = \mathcal{PE}_3 \cap \mathcal{PE}_4$ is closed in $\mathcal{PE}_3$, in $\mathcal{PE}_4$, and in $\mathcal{PE}$.*

Definition 10. *Consider wide subcategories $\mathcal{TPE}$, $\overline{\mathcal{QPE}}$, $\overline{\mathcal{SQPE}}$, $\overline{\mathcal{AQPE}}$, and $\overline{\mathcal{EPE}}$ of $\mathcal{PE}$, whose morphisms are of the following form:*

$$(t,x,u) \to \begin{cases} (t, y(t,x), v(t,x,u)) & \text{for } \mathcal{TPE} \\ (t, y(t,x), \varphi(t,x)u + \psi(t,x)) & \text{for } \overline{\mathcal{QPE}} \\ (t, y(x), \varphi(t,x)u + \psi(t,x)) & \text{for } \overline{\mathcal{SQPE}} \\ (t, y(x), \varphi(x)u + \psi(x)) & \text{for } \overline{\mathcal{AQPE}} \\ (t, y(x), u) & \text{for } \overline{\mathcal{EPE}} \end{cases}$$

Denote $\mathcal{TPE}_k = \mathcal{TPE} \cap \mathcal{PE}_k$.

Theorem 3.

1. *$\mathcal{TPE}$ is wide and plentiful in $\mathcal{PE}$.*
2. *$\mathcal{TPE}_k$ is closed in $\mathcal{TPE}$; it is wide and plentiful in $\mathcal{PE}_k$, $k = 1..5$.*

Definition 11. *Define the* ***category of quasilinear parabolic equations*** $\mathcal{QPE}$. $\mathcal{QPE}$ *is the following full subcategory of* $\overline{\mathcal{QPE}}$*: objects of* $\mathcal{QPE}$ *are equations of the form*

$$u_t = \sum_{i,j} b^{ij}(t,x,u)u_{ij} + \sum_i b^i\,(t,x,u)\,u_i + q(t,x,u), \tag{$\mathcal{QPE}$}$$

(in a local coordinates); morphisms of $\mathcal{QPE}$ *are maps of the form*

$$(t,x,u) \to (t, y(t,x), \varphi(t,x)u + \psi(t,x)).$$

Denote by $\mathcal{A}_{nc}(M,\Omega)$ the set of continuous positive functions $a\colon M\times\Omega\to\mathbb{R}$ that satisfies the condition

$$\forall m\in M \quad \exists u_1,u_2 \quad a(m,u_1)\neq a(m,u_2). \tag{$\mathcal{A}_{nc}$}$$

Define full subcategories of $\mathcal{QPE}$, whose objects are equations of the following form:

$$u_t = a(t,x,u)\sum_{i,j}\bar{b}^{ij}(t,x)u_{ij} + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{QPE}'$}$$

$$u_t = a(t,x,u)\sum_{i,j}\bar{b}^{ij}(t,x)u_{ij} + \sum_i b^i(t,x,u)u_i + q(t,x,u),\ a\in\mathcal{A}_{nc}(T\times X) \tag{$\mathcal{QPE}'_n$}$$

$$u_t = \sum_{i,j} b^{ij}(t,x)u_{ij} + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{QPE}'_1$}$$

$$u_t = a(t,x,u)\left(\sum_{i,j}\bar{b}^{ij}(t,x)u_{ij} + \sum_i \bar{b}^i(t,x)u_i\right) + \sum_i \xi^i(t,x)u_i + q(t,x,u) \tag{$\mathcal{QPE}''$}$$

$$u_t = a(t,x,u)\left(\sum_{i,j}\bar{b}^{ij}(t,x)u_{ij} + \sum_i \bar{b}^i(t,x)u_i\right) + q(t,x,u) \tag{$\mathcal{QPE}''_0$}$$

$$u_t = a(u)\left(\sum_{i,j}\bar{b}^{ij}(t,x)u_{ij} + \sum_i \bar{b}^i(t,x)u_i\right) + \sum_i \xi^i(t,x)u_i + q(t,x,u) \tag{$\mathcal{QPE}''_a(a)$}$$

$$u_t = \sum_{i,j} b^{ij}(t,x)u_{ij} + \sum_i b^i(t,x)u_i + q(t,x,u), \tag{$\mathcal{QPE}''_1$}$$

$$u_t = \sum_{i,j} b^{ij}(t,x)u_{ij} + \sum_i b^i(t,x)u_i + q_1(t,x)u + q_0(t,x), \tag{$\mathcal{QPE}''_{1q}$}$$

where $a>0$. The family of categories $\mathcal{QPE}''_a(a)$ is parameterized by functions $a(\cdot)$, that is one assigns the category $\mathcal{QPE}''_a(a)$ to each continuous positive function $a\colon\Omega\to\mathbb{R}$.

Furthermore, consider the full subcategory $\mathcal{QPE}_c$ of $\mathcal{QPE}$, whose objects are equations from $\mathcal{QPE}$ posed on a *compact* manifolds X.

Let us introduce the following notation for the intersections of enumerated "basic" subcategories: for a string σ we set $\mathcal{QPE}_\sigma = \cap\{\mathcal{QPE}_\alpha\colon \alpha\in\sigma\}$, $\mathcal{QPE}^\beta_\sigma = \mathcal{QPE}_\sigma\cap\mathcal{QPE}^\beta$. Particularly, $\mathcal{QPE}''_{0n}$ denotes the intersection $\mathcal{QPE}'_n\cap\mathcal{QPE}''_0$.

In the same way as in Remark 6, if we impose the condition $a(u_0)=1$, then we obtain the global function $a(u)$ for any equation from $\mathcal{QPE}''_a(a)$; such function $a(u)$ does not depend on the choice of neighborhood in $T\times X\times\Omega$ and on local coordinates.

Theorem 4.

1. $\mathcal{QPE}$ *is closed in* $\overline{\mathcal{QPE}}$ *and is fully dense in* $\mathcal{TPE}_1$.

2. $\mathcal{QPE}_c$ *is closed in* $\mathcal{QPE}$.

3. $\mathcal{QPE}' = \mathcal{QPE} \cap \mathcal{PE}_2 = \mathcal{QPE} \cap \mathcal{PE}_3$ *is fully dense in* $\mathcal{TPE}_3$ *and is closed in* $\mathcal{QPE}$.

4. $\mathcal{QPE}'_1 = \overline{\mathcal{QPE}} \cap \mathcal{PE}_5 = \mathcal{QPE}' \cap \mathcal{PE}_5$ *is fully dense in* $\mathcal{TPE}_5$ *and is closed in* $\mathcal{QPE}'$.

5. $\mathcal{QPE}''$ *is closed in* $\mathcal{QPE}'$.

6. $\mathcal{QPE}''_1 = \mathcal{QPE}'' \cap \mathcal{PE}_5 = \mathcal{QPE}'' \cap \mathcal{QPE}'_1 = \mathcal{QPE}''_a(1)$ *is closed in* $\mathcal{QPE}'_1$, *in* $\mathcal{QPE}''$, *and in* $\mathcal{QPE}''_0$.

7. $\mathcal{QPE}''_{1q}$ *is closed in* $\mathcal{QPE}''_1$.

8. $\mathcal{QPE}'_n$ *is closed in* $\mathcal{QPE}'$.

9. $\mathcal{QPE}''_{0n}$ *is fully plentiful in* $\mathcal{QPE}''_n$.

10. $\mathcal{QPE}''_{0c}$ *is fully dense in* $\mathcal{QPE}''_c$.

Let $\mathcal{A}_{\exp}$ be the set of functions of the form $a(u) = e^{\lambda u} H(u)$; let $\mathcal{A}_{\deg}$ be the set of functions of the form $a(u) = (u - u_0)^{\lambda} H(\ln(u - u_0))$, where λ, u_0 are arbitrary constants and $H(\cdot)$ is arbitrary nonconstant periodic function.

Theorem 5.

1. *If* $a \notin \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$, *then* $\mathcal{QPE}''_a(a)$ *is fully plentiful in* $\mathcal{QPE}''$.

2. $\mathcal{QPE}''_{0a}(a)$ *is fully plentiful in* $\mathcal{QPE}''_a(a)$; *if* $a \notin \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$, *then* $\mathcal{QPE}''_{0a}(a)$ *is fully plentiful in* $\mathcal{QPE}''_0$.

3. $\mathcal{QPE}''_{0ca}(a)$ *is fully dense in* $\mathcal{QPE}''_{ca}(a)$.

4. *Suppose* $\mathbf{A}$ *is an object of* $\mathcal{QPE}''_a(a)$, $F\colon \mathbf{A} \to \mathbf{B}$ *is a morphism in* $\mathcal{PE}$ *such that there is no object of* $\mathcal{QPE}''_a(a)$ *isomorphic to* $\mathbf{B}$ *in* $\mathcal{PE}$ *(that is* $a(\cdot) \in \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$*). Then there exist an object of* $\mathcal{QPE}''$ *isomorphic to* $\mathbf{B}$ *such that the composition of* $F\colon \mathbf{A} \to \mathbf{B}$ *with this isomorphism is of the form*

$$(t,x,u) \to \begin{cases} (t, y(t,x), u + \psi(t,x)), & a \in \mathcal{A}_{\exp} \\ (t, y(t,x), v_0 + (u - u_0)\exp(\psi(t,x))), & a \in \mathcal{A}_{\deg} \end{cases}$$

In addition, for each $t \in T, x_1, x_2 \in X$ *with* $y(t,x_1) = y(t,x_2)$ *the difference* $\psi(t,x_2) - \psi(t,x_1)$ *is an integral multiple of* $\hat{H}$, *where* $\hat{H}$ *is the period of function* H. *The same assertion holds if we replace* $\mathcal{QPE}''_a(a)$ *by* $\mathcal{QPE}''_{0a}(a)$ *and replace* $\mathcal{QPE}''$ *by* $\mathcal{QPE}''_0$.

Example 3. *Consider the equation*

$$E\colon u_t = (2 + \sin u)\, u_{xx}$$

of the category $\mathcal{QPE}''_{0a}(f)$, *where* $X = T = \Omega = \mathbb{R}$, $f(u) = 2 + \sin u$, $f \in \mathcal{A}_{\exp}$. *It admits both maps* $(t,x,u) \mapsto (t, x \bmod 2\pi, u)$ *and* $(t,x,u) \mapsto (t, x \bmod 2\pi, u + x)$. *In both*

cases $Y = S^1$. In the first case the quotient equation is of the form $v_t = (2 + \sin v)\, v_{yy}$ and is an object of $\mathcal{QPE}''_{0a}(f)$. In the second case the quotient equation is of the form $v_t = (2 + \sin(v + y))\, v_{yy}$; it is the object of $\mathcal{QPE}''_0$, but is not isomorphic to any object of $\mathcal{QPE}''_{0a}(f)$.

Definition 12. *The* ***category of semi-autonomous quasilinear parabolic equations*** *$\mathcal{SQPE}$ is the intersection $\overline{\mathcal{SQPE}} \cap \mathcal{QPE}''$. In other words, $\mathcal{SQPE}$ is the full subcategory of $\overline{\mathcal{SQPE}}$ and the wide subcategory of $\mathcal{QPE}''$, whose objects are equations of the form*

$$u_t = a(t,x,u)\left(\sum_{i,j} \bar{b}^{ij}(t,x)u_{ij} + \sum_i \bar{b}^i(t,x)u_i\right) + \sum_i \xi^i(t,x)u_i + q(t,x,u), \qquad (\mathcal{SQPE})$$

and morphisms are maps of the form $(t,x,u) \mapsto (t, y(x), \varphi(t,x)u + \psi(t,x))$.

Define the following full subcategories of $\mathcal{SQPE}$: $\mathcal{SQPE}_\sigma = \mathcal{SQPE} \cap \mathcal{QPE}''_\sigma$; $\mathcal{SQPE}_b$ is the category, whose objects are equations of the form

$$u_t = a(t,x,u)\left(\sum_{i,j} \bar{b}^{ij}(x)u_{ij} + \sum_i \bar{b}^i(t,x)u_i\right) + \sum_i \xi^i(t,x)u_i + q(t,x,u). \qquad (\mathcal{SQPE}_b)$$

Theorem 6.

1. *$\mathcal{SQPE}$ is closed in $\overline{\mathcal{SQPE}}$.*
2. *$\mathcal{SQPE}_0 = \overline{\mathcal{SQPE}} \cap \mathcal{QPE}''_0$, $\mathcal{SQPE}_n = \overline{\mathcal{SQPE}} \cap \mathcal{QPE}''_n$, and $\mathcal{SQPE}_b$ are closed in $\mathcal{SQPE}$.*
3. *$\mathcal{SQPE}_{0n}$ coincides with $\mathcal{QPE}''_{0n}$; it is closed in $\mathcal{SQPE}_0$ and in $\mathcal{SQPE}_n$.*
4. *$\mathcal{SQPE}_1 = \overline{\mathcal{SQPE}} \cap \mathcal{QPE}''_1 = \mathcal{SQPE}_a(1)$ is closed in $\mathcal{SQPE}_0$.*
5. *If $a \notin \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$, then $\mathcal{SQPE}_a(a)$ is fully plentiful in $\mathcal{SQPE}$.*

Definition 13. *The* ***category of autonomous quasilinear parabolic equations*** *$\mathcal{AQPE}$ is the full subcategory of $\overline{\mathcal{AQPE}}$, each object of $\mathcal{AQPE}$ is an equation of the form*

$$u_t = a(x,u)\,(\Delta u + \eta \nabla u) + \xi \nabla u + q(x,u) \qquad (\mathcal{AQPE})$$

posed on a Riemann manifold X equipped with vector fields ξ, η.
Define the following full subcategories of $\mathcal{AQPE}$: $\mathcal{AQPE}_\sigma = \mathcal{AQPE} \cap \mathcal{QPE}''_\sigma$ is the category, whose objects are equations of the form

$$\begin{aligned}
&u_t = a(x,u)\,(\Delta u + \eta\nabla u) + \xi\nabla u + q(x,u), \quad a \in \mathcal{A}_{nc}(X), && (\mathcal{AQPE}_n)\\
&u_t = a(x,u)(\Delta u + \eta\nabla u) + q(x,u), && (\mathcal{AQPE}_0)\\
&u_t = a(u)(\Delta u + \eta\nabla u) + \xi\nabla u + q(x,u), && (\mathcal{AQPE}_a(a))\\
&u_t = \Delta u + \xi\nabla u + q(x,u). && (\mathcal{AQPE}_1)
\end{aligned}$$

Theorem 7.

1. *$\mathcal{AQPE}$ is closed in $\overline{\mathcal{AQPE}}$.*

2. *$\mathcal{AQPE}_n$ is closed in $\mathcal{AQPE}$ and full in $\mathcal{SQPE}_{bn}$.*

3. *$\mathcal{AQPE}_0$ and $\mathcal{AQPE}_1$ are closed in $\mathcal{AQPE}$.*

4. *If $a(\cdot) \notin \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$, then $\mathcal{AQPE}_a(a)$ is fully plentiful in $\mathcal{AQPE}$.*

5. *$\mathcal{AQPE}_{na}(a)$ is closed in $\mathcal{SQPE}_{na}(a)$.*

Definition 14. *Define the following full subcategories of the category $\overline{\mathcal{EPE}}$ (its morphisms are a maps of the form $(t, x, u) \mapsto (t, y(x), u)$):*

$$\begin{aligned}\mathcal{EPE} &= \overline{\mathcal{EPE}} \cap \mathcal{AQPE},\\ \mathcal{EPE}_\sigma &= \overline{\mathcal{EPE}} \cap \mathcal{AQPE}_\sigma,\\ \mathcal{EPE}_a(a) &= \overline{\mathcal{EPE}} \cap \mathcal{AQPE}_a(a).\end{aligned}$$

Theorem 8.

1. *$\mathcal{EPE}$ is closed in $\overline{\mathcal{EPE}}$ and wide in $\mathcal{AQPE}$.*

2. *$\mathcal{EPE}_n$, $\mathcal{EPE}_0$, $\mathcal{EPE}_1$, and $\mathcal{EPE}_a(a)$ are closed in $\mathcal{EPE}$.*

3. *If $a \notin \mathcal{A}^{\text{ext}}_{\exp} \cup \mathcal{A}^{\text{ext}}_{\deg}$, then $\mathcal{EPE}_a(a)$ coincides with $\mathcal{AQPE}_a(a)$.*

Here $\mathcal{A}^{\text{ext}}_{\exp}$ is the set of functions $a(u)$ of the form $a(u) = e^{\lambda u} H(u)$, $\mathcal{A}^{\text{ext}}_{\deg}$ is the set of functions of the form $a(u) = (u - u_0)^\lambda H(\ln(u - u_0))$, λ, u_0 are arbitrary constants, $H(\cdot)$ is arbitrary periodic function (that is $\mathcal{A}_{\exp} \subset \mathcal{A}^{\text{ext}}_{\exp}$, $\mathcal{A}_{\deg} \subset \mathcal{A}^{\text{ext}}_{\deg}$).

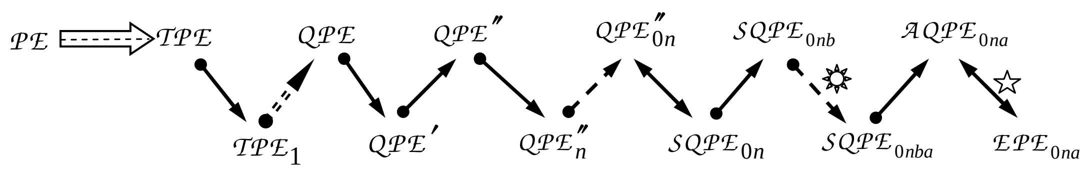

Figure 7: The sequence of arrows from $\mathcal{PE}$ to $\mathcal{EPE}_{0na}(a)$

Let us consider the sequence depicted on Fig. 7. Selecting the "weakest" arrow in this sequence, we obtain the following result.

Corollary 1.

1. *If $a \notin \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$, $a \neq \text{const}$, then $\mathcal{AQPE}_{0a}(a)$ is fully plentiful in $\mathcal{TPE}$ and plentiful in $\mathcal{PE}$.*

2. *If $a \notin \mathcal{A}^{\text{ext}}_{\exp} \cup \mathcal{A}^{\text{ext}}_{\deg}$, $a \neq \text{const}$, then $\mathcal{EPE}_{0a}(a)$ is fully plentiful in $\mathcal{TPE}$ and plentiful in $\mathcal{PE}$.*

8 Factorization of the reaction-diffusion equation

Let us consider a nonlinear reaction-diffusion equation

$$u_t = a(u)\,(\Delta u + \eta \nabla u) + q(x,u),$$

$a \neq \text{const}$, posed on a Riemann manifold X equipped with a vector field ξ. This equation defines the object $\mathbf{A}$ of the category $\mathcal{EPE}_{0na}(a)$. Using Corollary 1, we get the following result:

Corollary 2.

1. *If $a \notin \mathcal{A}_{\text{exp}} \cup \mathcal{A}_{\text{deg}}$, then for every morphism $F\colon \mathbf{A} \to \mathbf{B}$ of the category $\mathcal{PE}$ there exist isomorphism $I\colon \mathbf{B} \to \mathbf{B}'$ of the form* (1) *(in other words, bijective change of variables in the quotient equation), transforming F to the canonical morphism of the form*

$$(t,x,u) \to \begin{cases} (t, y(x), \varphi(x)u + \psi(x)) & \text{at } a \notin \mathcal{A}_{\text{exp}} \cup \mathcal{A}_{\text{deg}}; \\ (t, y(x), u) & \text{at } a \notin \mathcal{A}^{\text{ext}}_{\text{exp}} \cup \mathcal{A}^{\text{ext}}_{\text{deg}}. \end{cases} \tag{2}$$

   The corresponding canonical quotient equation B' posed on the Riemann manifold Y is of the form

$$v_t = a(v)\,(\Delta v + \mathrm{H}\nabla v) + Q\,(y,v)\,, \quad y \in Y, \quad \mathrm{H} \in TY \tag{3}$$

   with the same function a.

2. *If a morphism $F\colon \mathbf{A} \to \mathbf{B}$ of the category $\mathcal{TPE}$ transforms $\mathbf{A}$ to an equation of the form* (3)*, then F is of the form* (2)*.*

9 Comparison with the reduction by a symmetry group

As Remark 2 shows, our definition of morphism in $\mathcal{PDE}$ is a generalization of the reduction by a symmetry group. So we could obtain the sets of solutions being more common than the sets of group-invariant solutions of group analysis of PDE (though our approach is more laborious owing to the non-linearity of the system of PDE describing a morphisms). In what follows we show this on the example of a primitive morphism.

Definition 15. *A morphism $F\colon \mathbf{A} \to \mathbf{B}$ of a category $\mathcal{C}$ is called a* ***reducible in*** $\mathcal{C}$ *if in $\mathcal{C}$ there are exists non-invertible morphisms $G\colon \mathbf{A} \to \mathbf{C}$, $H\colon \mathbf{C} \to \mathbf{B}$ such that $F = H \circ G$. Otherwise, a morphism is called* ***primitive in*** $\mathcal{C}$.

Note that the reduction of PDE by a symmetry group defines a primitive morphism if and only if the group has no proper subgroups, i.e. the group is a discrete cyclic group of prime order. The reduction by any symmetry group that is not a discrete cyclic group of prime order may be always represented as a superposition of two nontrivial reductions, so the corresponding morphism is a superposition of two non-invertible morphisms and

therefore is reducible. In particular, this situation takes place for any nontrivial connected Lie group.

However, the situation for morphisms is completely different. Even a morphism that decreases the number of independent variables by 2 or more may be primitive; below we present an example of such a morphism. However, in the Lie group analysis we always have one-parameter subgroups of a symmetry group, so the morphism, corresponding to a symmetry group, is always reducible.

Example 4. *Consider the following morphism $F\colon \mathbf{A} \to \mathbf{B}$ in $\mathcal{PE}$:*

- $\mathbf{A}$ *is heat equation $u_t = a(u)\Delta u$ posed on $X = \{(x, y, z, w) : z < w\} \subset \mathbb{R}^4$ equipped with the metric*

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & \alpha & \beta \\ 0 & \alpha & 1 & 0 \\ 0 & \beta & 0 & 1 \end{pmatrix},$$

  where $\alpha = xe^w$, $\beta = xe^z$, $\gamma = 1 + \alpha^2 + \beta^2$, $a \notin \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$. In the coordinate form $\mathbf{A}$ *looks as*

$$a^{-1}(u)u_t = u_{xx} + u_{yy} - 2\alpha u_{yz} - 2\beta u_{yw} + \left(1 + \alpha^2\right) u_{zz} + \\ +2\alpha\beta u_{zw} + \left(1 + \beta^2\right) u_{ww} + (\alpha\beta)_w u_z + (\alpha\beta)_z u_w.$$

- $\mathbf{B}$ *is heat equation $a^{-1}(u)u_t = u_{xx} + u_{yy}$ posed on $Y = \{(x, y)\} = \mathbb{R}^2$ equipped with Euclidean metric.*

- *The morphism F is defined by the map $(t, (x, y, z, w), u) \mapsto (t, (x, y), u)$.*

This morphism decreases the number of independent variables by 2 and nevertheless is primitive in $\mathcal{PE}$.

Additional examples of morphisms that are not defined by any symmetry group of the given PDE, and also a detailed investigation of the case $\dim Y = \dim X - 1$, may be found in [Prokhorova, 2000], [Prokhorova, 2001].

10 Proofs

Proof of Theorem 1

Passing from the equation $u_t = Lu$ to the equation in the extended jet bundle for unknown submanifold $L \subset X \times T \times \Omega$ locally defined by the formula $f(t, x, u) = 0$, and expressing the derivatives of u by the corresponding derivatives of f, we obtain the following extended version of the equation E:

$$f_t f_u^2 = \sum_{i,j} b^{ij} \left(f_{ij} f_u^2 - (f_{iu} f_j + f_{ju} f_i) f_u + f_i f_j f_{uu}\right) - \sum_{i,j} c^{ij} f_i f_j f_u + \sum_i b^i f_i f_u^2 - q f_u^3 \quad (4)$$

Suppose $F\colon \mathbf{A} \to \mathbf{A}'$ is a morphism in $\mathcal{PE}$, $N' = X' \times T' \times \Omega'$, E' is defined by the equation

$$u' = \sum_{i',j'} B^{i'j'}(t', x', u')\, u'_{i'j'} + \sum_{i',j'} C^{i'j'}(t', x', u')\, u'_{i'} u'_{j'} + \sum_{i'} B^{i'}(t', x', u')\, u'_{i'} + Q(t', x', u').$$

Consider the extended analog of the last equation:

$$f'_{t'} f'^2_{u'} = \sum_{i',j'} B^{i'j'} \left(f'_{i'j'} f'^2_{u'} - \left(f'_{i'u'} f'_{j'} + f'_{j'u'} f'_{i'} \right) f'_{u'} + f'_{i'} f'_{j'} f'_{u'u'} \right) -$$
$$- \sum_{i',j'} C^{i'j'} f'_{i'} f'_{j'} f'_{u'} + \sum_{i'} B^{i'} f'_{i'} f'^2_{u'} - Q f'^3_{u'}, \quad (5)$$

where the equation $f'(t', x', u') = 0$ locally defines a submanifold L' of N'.

Recall that $F\colon (t, x, u) \mapsto (t', x', u')$ is morphism in $\mathcal{PE}$ if and only if for each point $\vartheta \in N$ and for each submanifold L' of N', $F(\vartheta) \in L'$, the following two conditions are equivalent:

- the 2-jet of L' at the point $F(\vartheta)$ satisfies (5)
- the 2-jet of $F^{-1}(L')$ at the point ϑ satisfies (4).

In other words, conditions "f' is solution of (5)" and "f is solution of (4)" must be equivalent when

$$f(t, x, u) = f'\left(t'(t, x, u), x'(t, x, u), u'(t, x, u)\right).$$

To find all such maps we use the following procedure:

1. Express derivatives of f in (4) through derivatives of f':
$$\frac{\partial f}{\partial t} = \frac{\partial f'}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial f'}{\partial x'^{i'}} \frac{\partial x'^{i'}}{\partial t} + \frac{\partial f'}{\partial u'} \frac{\partial u'}{\partial t}$$
and so on.

2. In the obtained identity substitute the combinations of the derivatives of f' for $\partial f'/\partial t'$ by formula (5). Then repeat this step for $\partial^2 f'/\partial t'^2$ in order to eliminate all derivatives with respect to t'. After reducing to common denominator, obtained identity will be of the form $\Phi = 0$, where Φ is rational function of partial derivatives of f' with respect to x' and u'. The coefficients $\phi_1, \ldots, \phi_s$ of Φ are functions of 4-jet of the map F.

3. Solve the system of PDEs $\phi_1 = 0, \ldots, \phi_s = 0$ for the map F.

Let us realize this procedure. Note that we shall not write out function Φ completely, but consider only some of its coefficients. We shall use the obtained information about F in order to simplify Φ step by step in the following manner.

First note that derivatives of the forth order arise only in the term $\partial^2 f'/\partial t'^2$ when we fulfill step 2 of the above procedure. Write this term before the final realization of step 2 for the sake of simplicity:

$$\Phi = \sum_{i,j} b^{ij} \left(t'_i t'_j f_u^2 - t'_i t'_u f_j f_u - t'_j t'_u f_i f_u + t'^2_u f_i f_j \right) \frac{\partial^2 f'}{\partial t'^2} + \ldots =$$
$$= \sum_{i,j} b^{ij} \left(t'_i f_u - t'_u f_i \right) \left(t'_j f_u - t'_u f_j \right) \frac{\partial^2 f'}{\partial t'^2} + \ldots$$

The coefficient at $\partial^2 f'/\partial t'^2$ must be zero, and the quadratic form b^{ij} is positive defined. We get ${t'}_i f_u = {t'}_u f_i$, so

$$ {t'}_i \left({f'}_{t'}{t'}_u + {f'}_{x'}{x'}_u + {f'}_{u'}{u'}_u\right) = {t'}_u \left({f'}_{t'}{t'}_i + {f'}_{x'}{x'}_i + {f'}_{u'}{u'}_i\right) $$

(here and below we use notations

$$ {f'}_{i'} = \frac{\partial f'}{\partial {x'}_{i'}}, \quad {f'}_{x'}{x'}_u = \sum_{j'} {f'}_{j'}{x'}^{j'}_u $$

and so on). Hence we obtain the following system of equations:

$$ \begin{cases} {t'}_u {u'}_i = {u'}_u {t'}_i \\ {t'}_u {x'}_i = {x'}_u {t'}_i \end{cases} \tag{6} $$

One of the following three conditions holds:

1. ${t'}_u = 0$, ${t'}_x = 0$;
2. ${t'}_u = 0$, ${t'}_x \neq 0$;
3. ${t'}_u \neq 0$.

In the second case ${u'}_u = {x'}_u = 0$. Taking into account equality ${t'}_u = 0$, we obtain a desired contradiction to the assumption that F is a submersion.

In the third case we get from (6) ${t'}_x = \omega {t'}_u$, ${u'}_x = \omega {u'}_u$, ${x'}^{i'}_x = \omega {x'}^{i'}_u$, where $\omega = {t'}_x/{t'}_u \in \Gamma\left(\pi^* T^* X\right)$, $\pi\colon N = T \times X \times \Omega \to X$ is the natural projection, $\pi^* T^* X$ is the vector bundle over N induced by π from the cotangent bundle T^*X over X, $\omega = \sum_i \omega_i(t,x,u)\mathrm{d}x^i$ in local coordinates. This implies that $f_x = \omega f_u$. Substituting last formula to (4), we get

$$ f_t = f_u \left[\sum_{i,j} b^{ij} \left(\frac{\partial \omega_i}{\partial x^j} - \omega_j \frac{\partial \omega_i}{\partial u} \right) - \sum_{i,j} c^{ij}\omega_i\omega_j + \sum_i b^i \omega_i - q \right]. $$

Denote by $\zeta\left(t,x,u\right)$ the expression in square brackets. Then $f_t = \zeta f_u$. Expressing derivatives of f in terms of derivatives of f', we get ${t'}_t = \zeta {t'}_u$, ${x'}_t = \zeta {x'}_u$, ${u'}_t = \zeta {u'}_u$. Consider the field of hyperplanes that kill the 1-form $\mathrm{d}t'$ in the tangent bundle TM (recall that ${t'}_u \neq 0$, so $\mathrm{d}t'$ is nondegenerated). The differential of the map F vanishes on these hyperplanes, because $\mathrm{d}u' \wedge \mathrm{d}t' = \mathrm{d}x'^{i'} \wedge \mathrm{d}t' = 0$. Therefore $\mathrm{rang}(\mathrm{d}F) \leq 1$ and F could not be submersive, because $\dim N' \geq 3$.

Finally, only the first case is possible. Hence t' is a function of t, and ${f'}_{t'}$ could appear only in the representation of f_t. Let us look at the terms of Φ, containing $\left({f'}_{u'}\right)^{-2}$:

$$ \Phi = \sum_{i',j',k',l'} {t'}_t {x'}^{i'}_u {x'}^{j'}_u B'^{k'l'} {f'}_{i'}{f'}_{j'}{f'}_{k'}{f'}_{l'}{f'}_{u'u'}\left({f'}_{u'}\right)^{-2} + \dots . $$

Substitution of any covector $\omega = \sum_{i'} \omega_{i'} \mathrm{d}x'^{i'} \in \Gamma\left(T^* X'\right)$ to the expression

$$ \sum_{i',j',k',l'} {t'}_t {x'}^{i'}_u {x'}^{j'}_u B'^{k'l'} \omega_{i'}\omega_{j'}\omega_{k'}\omega_{l'} = {t'}_t \left(\sum_{i'} {x'}^{i'}_u \omega_{i'} \right)^2 \left(\sum_{k',l'} B'^{k'l'} \omega_{k'}\omega_{l'} \right) $$

should give zero. The form $B'^{k'l'}$ is positively defined so $\sum_{k',l'} B'^{k'l'}\omega_{k'}\omega_{l'} > 0$ when $\omega \neq 0$. Taking into account that F is submersive, we obtain ${t'}_t \neq 0$, so $\sum_{i'} {x'}^{i'}_u \omega_{i'} = 0$ for any ω, that is ${x'}_u \equiv 0$. Hence $x' = x'\left(t,x\right)$, $t' = t'\left(t\right)$. $\square$

Proof of Theorem 2

The map $(t,x,u)\mapsto(\tau(t),y\left(t,x\right),v(t,x,u))$ is a morphism in $\mathcal{PE}$ if and only if

$$\begin{cases}\tau_t B^{kl}=\sum\limits_{i,j} b^{ij}y_i^k y_j^l\\ \tau_t C^{kl}=\left(\ln U_v\right)_v B^{kl}+U_v\sum\limits_{i,j} c^{ij}y_i^k y_j^l\\ \tau_t B^k=\sum\limits_{i,j} b^{ij}y_{ij}^k+2\sum\limits_{i,j} b^{ij}\left(\ln U_v\right)_j y_i^k+2\sum\limits_{i,j} c^{ij}U_j y_i^k+\sum\limits_{i} b^i y_i^k-y_t^k\\ \tau_t Q=U_v^{-1}\left(\sum\limits_{i,j} b^{ij}U_{ij}+\sum\limits_{i,j} c^{ij}U_iU_j+\sum\limits_{i} b^iU_i+q(t,x,U)-U_t\right)\end{cases}\tag{7}$$

where function $u=U\left(t,x,v\right)$ is the inverse of the $v\left(t,x,u\right)$. The quotient equation is written as $v_\tau=\sum_{k,l}B^{kl}v_{kl}+\sum_{k,l}C^{kl}v_kv_l+\sum_k B^kv_k+Q$. Here and below indexes i, j relate to x, indexes k, l relate to y.

By definition, all $\mathcal{PE}_k$ are full subcategories of $\mathcal{PE}$.

1. Let us prove that $\mathcal{PE}_1$ is closed in $\mathcal{PE}$. Suppose $\mathbf{A}\in\mathrm{Ob}_{\mathcal{PE}_1}$, $F\colon\mathbf{A}\to\mathbf{B}$ is a morphism in $\mathcal{PE}$. Then $c^{ij}=\lambda(t,x,u)b^{ij}$. From the second equation of system (7) we get

$$C^{kl}\left(\tau,y,v\right)=B^{kl}\left(\tau,y,v\right)\left[\tau_t^{-1}\left(\ln U_v\right)_v+\lambda\left(t,x,u\right)U_v\right].$$

The quadratic form B^{kl} is nondegenerated at any point (τ,y,v), so expression in square brackets is function of (τ,y,v): $\tau_t^{-1}\left(\ln U_v\right)_v+\lambda(t,x,u)U_v=\Lambda\left(\tau,y,v\right)$, and $C^{kl}\left(\tau,y,v\right)=\Lambda\left(\tau,y,v\right)B^{kl}(\tau,y,v)$. Thus $\mathbf{B}\in\mathrm{Ob}_{\mathcal{PE}_1}$.

Let us show that $\mathcal{PE}_2$ is closed in $\mathcal{PE}$. Suppose $\mathbf{A}\in\mathrm{Ob}_{\mathcal{PE}_2}$, $F\colon\mathbf{A}\to\mathbf{B}$ is a morphism in $\mathcal{PE}$. Then $b^{ij}=a(t,x,u)\bar{b}^{ij}(t,x)$. Using the first equation of the system (7), we receive

$$\tau_tB^{kl}=a(t,x,u)\left(\sum_{i,j}\bar{b}^{ij}y_i^ky_j^l\right)_{(t,x)}.$$

Taking into account that the quadratic form B^{kl} is nondegenerated, we obtain that $B^{11}\neq 0$ everywhere. From the equality

$$\frac{B^{kl}}{B^{11}}\left(\tau,y,v\right)=\frac{\sum_{i,j}\bar{b}^{ij}y_i^ky_j^l}{\sum_{i,j}\bar{b}^{ij}y_i^1y_j^1}\left(t,x\right)$$

we receive that this fraction is function of (t,y). Thus

$$B^{kl}(\tau,y,v)=A(\tau,y,v)\bar{B}^{kl}\left(\tau,y\right)$$

for $A\left(\tau,y,v\right)=B^{11}\left(\tau,y,v\right)$ and some functions $\bar{B}^{kl}(t,y)$. Therefore, $\mathbf{B}\in\mathrm{Ob}_{\mathcal{PE}_2}$. □

2. $\mathcal{PE}_3=\mathcal{PE}_1\cap\mathcal{PE}_2$ is closed in $\mathcal{PE}$, in $\mathcal{PE}_1$, and in $\mathcal{PE}_2$, because $\mathcal{PE}_1$ and $\mathcal{PE}_2$ are closed in $\mathcal{PE}$. □

3. Suppose $\mathbf{A}\in\mathrm{Ob}_{\mathcal{PE}_4}$ and $F\colon\mathbf{A}\to\mathbf{B}$ is a morphism of $\mathcal{PE}$. From the first equation of (7) we obtain that $B^{kl}\left(\tau,y,v\right)$ is independent of v. Hence $B^{kl}=B^{kl}\left(\tau,y\right)$, $\mathcal{PE}_4$ is closed in $\mathcal{PE}$, so it is closed in $\mathcal{PE}_2$ too. □

4. Since $\mathcal{PE}_3$ and $\mathcal{PE}_4$ are closed in $\mathcal{PE}$, we obtain that $\mathcal{PE}_5=\mathcal{PE}_3\cap\mathcal{PE}_4$ is closed in $\mathcal{PE}$, $\mathcal{PE}_3$ and $\mathcal{PE}_4$. □

Proof of Theorem 3

1. By definition, $\mathcal{TPE}$ is wide in $\mathcal{PE}$.
Suppose $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{PE}$. By Theorem 1, the function $\tau(t)$ is nondegenerated, so we could consider the inverse function $t(\tau)$. The map $(\tau, y, v) \to (t(\tau), y, v)$ is an isomorphism in $\mathcal{PE}$. Note that superposition of F with this isomorphism is a morphism in $\mathcal{TPE}$. Therefore $\mathcal{TPE}$ is plentiful in $\mathcal{PE}$. $\square$

2. $\mathcal{TPE}_k$ is closed in $\mathcal{PE}$, and $\mathcal{TPE}$ is wide and plentiful in $\mathcal{PE}$. Thus $\mathcal{TPE}_k = \mathcal{PE}_k \cap \mathcal{TPE}$ is closed in $\mathcal{TPE}$ and also it is wide and plentiful in $\mathcal{PE}_k$. $\square$

Proof of Theorem 4

Using the system (7), we obtain that the map $(t, x, u) \to (t, y, \varphi u + \psi)$ is a morphism in $\mathcal{QPE}$ if and only if

$$\begin{cases} B^{kl} = \sum\limits_{i,j} b^{ij} y_i^k y_j^l \\ B^k = \sum\limits_{i,j} b^{ij} y_{ij}^k + 2 \sum\limits_{i,j} b^{ij} (\ln \bar{\varphi})_j \, y_i^k + \sum\limits_{i} b^i y_i^k - y_t^k \\ Q\bar{\varphi} = \left(\sum\limits_{i,j} b^{ij} \bar{\varphi}_{ij} + \sum\limits_{i} b^i \bar{\varphi}_i - \bar{\varphi}_t \right) v + \left(\sum\limits_{i,j} b^{ij} \bar{\psi}_{ij} + \sum\limits_{i} b^i \bar{\psi}_i - \bar{\psi}_t \right) + q\left(t, x, \bar{\varphi} v + \bar{\psi}\right) \end{cases}, \tag{8}$$

where $\bar{\varphi} = \varphi^{-1}$, $\bar{\psi} = -\varphi^{-1}\psi$, so $U = \bar{\varphi} v + \bar{\psi}$. By definition, all subcategories of $\overline{\mathcal{QPE}}$ considered in the Theorem are full subcategories of $\overline{\mathcal{QPE}}$.

1a. If $c^{ij} = 0$ and v is linear in u, then $C^{kl} = 0$. By the second equation of the system (7), it follows that $\mathcal{QPE}$ is closed in $\overline{\mathcal{QPE}}$.

1b. Let $F\colon \mathbf{A} \to \mathbf{B}$, $(t, x, u) \mapsto (t, y(t, x), v(t, x, u))$ be a morphism in $\mathcal{TPE}_1$, and $\mathbf{A}, \mathbf{B} \in \mathrm{Ob}_{\mathcal{QPE}}$. Using the second equation of the system (7), we get $(\ln U_v)_v B^{kl} = C^{kl} = 0$. It follows that U is linear in v, v is linear in u, F is a morphism in $\mathcal{QPE}$, and $\mathcal{QPE}$ is full in $\mathcal{TPE}$.

1c. Suppose $\mathbf{A} \in \mathrm{Ob}_{\mathcal{TPE}_1}$. Fix $u_0 \in \Omega_{\mathbf{A}}$ and consider the map $F\colon (t, x, u) \mapsto (t, x, v(t, x, u))$,

$$v(t, x, u) = \int\limits_{u_0}^{u} \exp \left(\int\limits_{u_0}^{\xi} \lambda(t, x, \varsigma)\, d\varsigma \right) d\xi.$$

F define the isomorphism in $\mathcal{TPE}_1$ from $\mathbf{A}$ to $\mathbf{B}$ with

$$C^{ij} = (\ln U_v)_v \, b^{ij} + U_v \lambda b^{ij} = v_u^{-1} \left(\lambda - (\ln v_u)_u\right) = 0.$$

Therefore every object of $\mathcal{TPE}_1$ is isomorphic in $\mathcal{TPE}_1$ to some object of $\mathcal{QPE}$, and $\mathcal{QPE}$ is full in $\mathcal{TPE}_1$. $\square$

2. The image of a compact under a continuous map is compact. The surjectivity of the map completes the proof. $\square$

3. $\mathcal{TPE}_3$ is closed in $\mathcal{PE}_1$, $\mathcal{QPE}$ is fully dense in $\mathcal{PE}_1$. $\square$

4. $\mathcal{TPE}_5$ is closed in $\mathcal{TPE}_3$, and $\mathcal{QPE}'$ is fully dense in $\mathcal{TPE}_3$. Equality $\mathcal{QPE}'_1 = \mathcal{TPE}_5 \cap \mathcal{QPE}'$ concludes the proof.

5. Let $\mathbf{A} \in \mathrm{Ob}_{\mathcal{QPE}''}$, and $F\colon \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathcal{QPE}'$. From the first equation of the system (8) we obtain

$$a(t,x,u) = A(t,y,v)\bar{a}(t,x), \tag{9}$$

where $\bar{a}(t,x) = B^{11}\left(t, y\left(t,x\right)\right) \Big/ \left(\sum_{i,j} b^{ij}(t,x) y_i^1 y_j^1 (t,x)\right)$.

From the second equation of (8) we obtain

$$B^k(t,y,v) = A(t,y,v)\omega^k\left(t,x\right) + \mu^k(t,x), \tag{10}$$

where

$$\omega^k(t,x) = \bar{a}\left(\sum_{i,j} \bar{b}^{ij} y_{ij}^k + 2\sum_{i,j} \bar{b}^{ij}\left(\ln\bar{\varphi}\right)_j y_i^k + \sum_i \bar{b}^i y_i^k\right), \quad \mu^k(t,x) = \sum_i \xi^i y_i^k - y_t^k.$$

Further we need the following statement.

Lemma 5 (about the extension of a function)**.** *Suppose M, N are C^r-manifolds, $1 \le r \le \infty$, $F\colon M \to N$ is a surjective C^r-submersion, $\mu\colon M \to \mathbb{R}$ is a C^s-function, $0 \le s \le r$ (if $s = 0$ then μ is continuous). Take*

$$N_0 = \left\{ n \in N\colon \quad \mu|_{F^{-1}(n)} = \mathrm{const} \right\},$$

$$M_0 = F^{-1}\left(N_0\right) = \left\{ m \in M\colon\ \forall m' \in M\ \left[F\left(m'\right) = F\left(m\right)\right] \Rightarrow \left[\mu\left(m'\right) = \mu\left(m\right)\right]\right\},$$

$F_0 = F|_{M_0}$, $\mu_0 = \mu|_{M_0}$, and define the function $\nu_0\colon N_0 \to \mathbb{R}$ by the formula $\nu_0 F_0 = \mu_0$ (see Fig. 8(a)). Then ν_0 can be extended from N_0 to the entire manifold N so that the extended function $\nu\colon N \to \mathbb{R}$ will have class C^s of smoothness (see Fig. 8(b); both diagrams Fig. 8(a, b) are commutative).

$$\begin{tikzcd} M_0 \arrow[rr, "F_0"] \arrow[dd, hook] \arrow[dr, "\mu_0"'] & & N_0 \arrow[dd, hook] \arrow[dl, "\nu_0"] \\ & \mathbb{R} & \\ M \arrow[ur, "\mu"] \arrow[rr, two heads, "F"] & & N \end{tikzcd}
\qquad
\begin{tikzcd} M_0 \arrow[rr, "F_0"] \arrow[dd, hook] \arrow[dr, "\mu_0"'] & & N_0 \arrow[dd, hook] \arrow[dl, "\nu_0"] \\ & \mathbb{R} & \\ M \arrow[ur, "\mu"] & & N \arrow[ul, "\nu"'] \end{tikzcd}$$

a *b*

Figure 8: The extension of a function

Proof of Lemma 5

Take an open covering $\{V_i\colon i \in I\}$ of N such that for every V_i there exist a C^r-smooth section $p_i\colon V_i \to M$ over V_i, $F \circ p_i = \mathrm{id}|_{V_i}$ (such a covering exists, because F is submersive and surjective). There exist a C^r-partition of unity $\{\lambda_i\}$ that is subordinated to $\{V_i\}$ [Hirsch, 1976]. Define the functions

$$\nu_i(n) = \begin{cases} \lambda_i(n)\,\mu(p_i(n)), & n \in V_i \\ 0, & n \notin V_i \end{cases}.$$

Then $\nu(n) = \sum\limits_{i \in I} \nu_i(n)$ is the desired function. □

Proof of Theorem 4 (continuation)

Fix k. In the notations and assumption of Lemma 5, let us replace the map F by $(t,x) \mapsto (t, y(t,x))$ and the continuous function μ by $\mu^k(t,x)$. We obtain that there exists a continuous function $\nu^k(t,y)$ such that for each (t_0, y_0) if $\mu^k(t,x)$ is constant on the pre-image of (t_0,y_0) with respect to the map $(t,x) \mapsto (t, y(t,x))$ then $\nu^k(t_0,y_0)$ coincides with this constant. Denote

$$\bar{B}^k(t,y,v) = \left(B^k(t,y,v) - \nu^k(t,y)\right)/A(t,y,v). \tag{11}$$

Let us consider the following two cases for every point (t_0, y_0).

Case 1: $A(t_0,y_0,v)$ is independent of v. Using (10), we obtain that $B^k(t_0,y_0,v)$ is independent of v; and using (11) that $\bar{B}^k$ is independent of v.

Case 2: For given (t_0,y_0) the set $\{A(t_0,y_0,v) : v \in \Omega\}$ contains more then one element. Using (10), we obtain that the restriction of $\mu^k(t_0,x)$ to the pre-image of the point (t_0,y_0) is constant. Then $\mu^k(t_0,x) = \nu^k(t_0,y_0)$ on this pre-image, and $\bar{B}^k = \omega^k(t,x)$ is independent of v in this case too.

Therefore, $B^k(t,y,v) = A(t,y,v)\bar{B}^k(t,y) + \nu^k(t,y)$. So, the equation **B** is of the form

$$v_t = A(t,y,v)\left(\sum_{k,l} \bar{B}^{kl}(t,y)v_{kl} + \sum_k \bar{B}^k(t,y)v_k\right) + \sum_k \nu^k(t,y)v_k + Q(t,y,v),$$

and **B** is the object of $\mathcal{QPE}''$.

For F to be a morphism in $\mathcal{QPE}''$, it is necessary and sufficient to have

$$\begin{cases} a(t,x,u) = A(t,y,v)\bar{a}(t,x) \\ \bar{B}^{kl}(t,y) = \bar{a}\sum\limits_{i,j} \bar{b}^{ij} y_i^k y_j^l(t,x) \\ y_t^k + \Xi^k - \sum\limits_i \xi^i y_i^k = a(t,x,u)\left(\sum\limits_{i,j} \bar{b}^{ij} y_{ij}^k + 2\sum\limits_{i,j} \bar{b}^{ij}(\ln\bar{\varphi})_j\, y_i^k + \sum\limits_i \bar{b}^i y_i^k - B^k/\bar{a}\right) \\ Q\bar{\varphi} = \left(\sum\limits_{i,j} a\bar{b}^{ij}\bar{\varphi}_{ij} + \sum\limits_i \left(a\bar{b}^i + \xi_i\right)\bar{\varphi}_i - \bar{\varphi}_t\right) v + \\ \qquad + \left(\sum\limits_{i,j} a\bar{b}^{ij}\bar{\psi}_{ij} + \sum\limits_i \left(a\bar{b}^i + \xi_i\right)\bar{\psi}_i - \bar{\psi}_t\right) + q\left(t,x,\bar{\varphi}v + \bar{\psi}\right) \end{cases} \tag{12}$$

6. $\mathcal{QPE}_1''$ is closed in $\mathcal{QPE}''$ and in $\mathcal{QPE}_1'$, because $\mathcal{QPE}''$ and $\mathcal{QPE}_1'$ are closed in $\mathcal{QPE}'$. $\mathcal{QPE}_1''$ is closed in $\mathcal{QPE}_0''$, because $\mathcal{QPE}_0''$ is the subcategory of $\mathcal{QPE}''$. $\square$

7. Suppose $\mathbf{A} \in \mathrm{Ob}_{\mathcal{QPE}_{1q}''}$, $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{QPE}_1''$. From the third equation of (8) we get

$$Q(t, y, v) =$$

$$= \left(\sum_{i,j} b^{ij} \bar{\varphi}_{ij} + \sum_i b^i \bar{\varphi}_i + q_1(t, x) - \bar{\varphi}_t \right) \bar{\varphi}^{-1} v + \left(\sum_{i,j} b^{ij} \bar{\psi}_{ij} + \sum_i b^i \bar{\psi}_i + q_0(t, x) - \bar{\psi}_t \right) \bar{\varphi}^{-1} =$$

$$= Q_1(t, x) v + Q_0(t, x),$$

so Q_1, Q_0 are functions of (t, y), and $\mathbf{B} \in \mathrm{Ob}_{\mathcal{QPE}_{1q}''}$. Thus $\mathcal{QPE}_{1q}''$ is closed in $\mathcal{QPE}_1''$. $\square$

8. Suppose $\mathbf{A} \in \mathrm{Ob}_{\mathcal{QPE}_n'}$, $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{QPE}'$. For given (t_0, y_0) let us fix arbitrary x_0 such that $y\,(t_0, x_0) = y_0$. Using $\bar{\varphi} \neq 0$ and $a \in \mathcal{A}_{nc}\,(T \times X)$, from (9) we get

$$A\,(t_0, y_0, v) = a\left(t_0, x_0, \bar{\varphi}\,(t_0, x_0)\, v + \bar{\psi}\,(t_0, x_0)\right) \bar{a}\,(t_0, x_0) \neq \text{const}.$$

Finally, we obtain $A \in \mathcal{A}_{nc}\,(T \times Y)$, and $\mathbf{B} \in \mathrm{Ob}_{\mathcal{QPE}_n'}$, so $\mathcal{QPE}_n'$ is closed in $\mathcal{QPE}'$. $\square$

9. Suppose $\mathbf{A} \in \mathrm{Ob}_{\mathcal{QPE}_{0n}''}$, $\mathbf{B} \in \mathrm{Ob}_{\mathcal{QPE}_n''}$. Substituting $\xi_i = 0$ in the third equation of (12), we get

$$y_t^k + \Xi^k(t, y) = a(t, x, u) \left(\sum_{i,j} \bar{b}^{ij} y_{ij}^k + 2 \sum_{i,j} \bar{b}^{ij} \left(\ln \bar{\varphi}\right)_j y_i^k + \sum_i \bar{b}^i y_i^k - B^k / \bar{a} \right) (t, x).$$

Since $a \in \mathcal{A}_{nc}(T \times X)$, and left hand side is independent of u, it follows that both sides of this equality vanishes, and

$$y_t^k = -\Xi^k(t, y) \tag{13}$$

The function $y(t, x)$ satisfies the ordinary differential equation (13) with smooth right hand side, so for any t, t' an equality $y(t, x_1) = y(t, x_2)$ implies that $y(t', x_1) = y(t', x_2)$. Let 1-parameter transformation group $g_s\colon T \times Y \to T \times Y$ be given by $(t, y(t, x)) \mapsto (t{+}s, y(t{+}s, x))$. This group is correctly defined when $T = \mathbb{R}$; otherwise transformations g_s are partially defined, nevertheless reasoning below remains correct after small refinement.

For every s the composition $g_s g_{-s}$ is identity, so g_s is bijective. $\{g_s\}$ is a flow map of the smooth vector field $\partial_t - \sum_k \Xi^k(t, y) \partial_{y^k}$, so transformations $\{g_s\}$ are smooth by both t and y.

Define the map $z(t, y)$ by the equality $g_{-t}(t, y) = (0, z(t, y))$. Then the map $G\colon T \times Y \to T \times Y$, $(t, y) \mapsto (t, z(t, y))$ is the isomorphism in $\mathcal{QPE}''$ such that for every x, t $z(t, y(t, x)) = z(0, y(0, x))$. Therefore $G \circ F \in \mathrm{Hom}_{\mathcal{QPE}_0''}$. $\square$

10. Let $\mathbf{A}$ be an object of $\mathcal{QPE}_c''$. Consider the solution $y\colon T \times X \to X$ of the linear PDE $\partial y^k / \partial t = \sum_i \xi^i(t, x) \partial y^k / \partial x^i$ (the solution exists in light of the compactness of X). The isomorphism $(t, x, u) \mapsto (t, y(t, x), u)$ maps $\mathbf{A}$ to some object of $\mathcal{QPE}_0''$. Thus $\mathcal{QPE}_{0c}''$ is closed in $\mathcal{QPE}_c''$. $\square$

Proof of Theorem 5

If $a \neq \text{const}$, then $\mathcal{QPE}''_{0a}(a)$ is fully plentiful in $\mathcal{QPE}''_a(a)$ thanks to the part 9 of Theorem 4.

If $a = \text{const}$, then $\mathcal{QPE}''_a(a)$ coincides with $\mathcal{QPE}''_1$, which is closed in $\mathcal{QPE}''$ by the Theorem 4. So $\mathcal{QPE}''_a(a)$ is fully plentiful in $\mathcal{QPE}''$.

Suppose now that $a \neq \text{const}$, $\mathbf{A} \in \mathrm{Ob}_{\mathcal{QPE}''_a(a)}$, and $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{QPE}''$. Consider the equation (9) as functional one:

$$a\left(\bar{\varphi}(t,x)v + \bar{\psi}\,(t,x)\right) = A(t,y,v)\bar{a}(t,x). \tag{14}$$

Let us consider the following three cases.

<u>Case 1.</u> $a(u) = He^{\lambda u}$, $\lambda, H = \text{const}$, $\lambda \neq 0$. Substituting the formula for a to (14), we get $\lambda\bar{\varphi}(t,x)v - \ln A(t,y,v) = \left(\ln\bar{a} - \lambda\bar{\psi} - \ln H\right)$. The right hand side of the equality is a function of (t,x), so $\bar{\varphi} = \bar{\varphi}(t,y)$, and the isomorphism $(t,y,v) \mapsto (t,y,\bar{\varphi}(t,y)v)$ maps $\mathbf{B}$ to some object of $\mathcal{QPE}''_a(a)$.

<u>Case 2.</u> $a(u) = H\left(u - u_0\right)^{\lambda}$, $\lambda, H, u_0 = \text{const}$, $\lambda \neq 0$. Substituting the formula for a to (14), we get

$$\left(v + \bar{\varphi}^{-1}(t,x)\left(\bar{\psi}(t,x) - u_0\right)\right)^{\lambda} = A(t,y,v)H^{-1}\bar{\varphi}^{-\lambda}\bar{a}(t,x).$$

Thus $\bar{\varphi}^{-1}\left(\bar{\psi} - u_0\right) = q(t,y)$ for some function q, and the object $\mathbf{B}$ maps by the isomorphism $(t,y,v) \mapsto (t,y,v + q(t,y) + u_0)$ to some object of $\mathcal{QPE}''_a(a)$.

<u>Case 3.</u> Suppose now that $a(u)$ is neither $He^{\lambda u}$ nor $H\left(u - u_0\right)^{\lambda}$. Denote $\bar{x} = (t,x)$, $\bar{y} = (t,y)$, $\alpha = \ln a$. Fix a point $\bar{y}_0$ and take $Z = \{\bar{x}\colon \bar{y}\left(\bar{x}\right) = \bar{y}_0\} \subset T \times X$. Using (14), we obtain that $\forall \bar{x}_0, \bar{x}_1 \in Z \quad \alpha\left(\bar{\varphi}_1 z + \bar{\psi}_1\right) - \alpha\left(\bar{\varphi}_0 z + \bar{\psi}_0\right)$ is independent of v, where $\bar{\varphi}_i = \bar{\varphi}\left(\bar{x}_i\right)$, $\bar{\psi}_i = \bar{\psi}\left(\bar{x}_i\right)$). Consider additive subgroup $G = G\left(\bar{y}_0\right)$ of $\mathbb{R}$ generated by the set $\left\{\ln\bar{\varphi}\left(\bar{x}\right) - \ln\bar{\varphi}\left(\bar{x}_0\right) : \bar{x} \in Z\right\}$.

Consider the following two subcases.

<u>Case 3.1.</u> $G \neq \{0\}$.
Put $\hat{H}_1 = \ln\bar{\varphi}_1 - \ln\bar{\varphi}_0 \in G - \{0\}$, $u_0 = \left(\bar{\psi}_0 - \bar{\psi}_1\right)/\left(\bar{\varphi}_1 - \bar{\varphi}_0\right)$. Substituting $v = \left(w + u_0 - \bar{\psi}_0\right)/\bar{\varphi}_0$, for any w we have $\alpha\left(e^{\hat{H}_1}w + u_0\right) - \alpha\left(w + u_0\right) = c = \text{const}$. Consider the function $\beta\left(x\right) = \alpha\left(e^{x} + u_0\right)$. Using $\beta\left(x + \hat{H}_1\right) = \beta(x) + c$, we obtain that the function $\beta\left(x\right) - \lambda x$ is $\hat{H}_1$-periodic, where $\lambda = c/\hat{H}_1$. Then

$$a(u) = \left(u - u_0\right)^{\lambda} H\left(\ln\left(u - u_0\right)\right),$$

where H is $\hat{H}_1$-periodic, $H \neq \text{const}$, because case "$H = \text{const}$" was already considered. Let $\hat{H} > 0$ be the smallest positive period of H. For all $\bar{x} \in Z$ the number $\ln\bar{\varphi}\left(\bar{x}\right) - \ln\bar{\varphi}_0$ is a multiple of $\hat{H}$, so for any $\bar{y}_0$ we have $\bar{\varphi}\left(\bar{x}\right) \in \left\{\bar{\varphi}_0 \exp\left(k\hat{H}\right) : k \in \mathbb{Z}\right\}$. Note that $\hat{H}$ is independent of $\bar{y}_0$, because $a(u)$ is independent of $\bar{y}_0$.

<u>Case 3.2.</u> $G = \{0\}$, that is $\bar{\varphi}|_Z \equiv \bar{\varphi}_0 = \text{const}$. Now we have the following two subsubcases:

Case 3.2.a. $\bar{\psi}\big|_Z \neq$ const, that is $\exists \bar{x}_0, \bar{x}_1 \in Z \colon \bar{\psi}(\bar{x}_1) - \bar{\psi}(\bar{x}_0) = \hat{H}_1 \neq 0$. Then $\alpha\left(u + \hat{H}_1\right) - \alpha(u) =$ const. By the same token as in case 3.1 we get $a(u) = H(u)e^{\lambda u}$, where $\lambda =$ const, H is a periodic function with the smallest period $\hat{H} > 0$. Note that such representation of $a(u)$ is unique. Substituting this to (14), we obtain that $\forall \bar{y}\ \forall \bar{x}_0, \bar{x}_1 \in Z_{\bar{y}}$ the number $\bar{\psi}(\bar{x}_1) - \bar{\psi}(\bar{x}_0)$ is a multiple of $\hat{H}$.

Case 3.2.b. $\bar{\psi}\big|_Z =$ const for given $\bar{y}_0$. We already considered thes cases $a(u) = H(u)e^{\lambda u}$ and $a(u) = (u - u_0)^\lambda H(\ln(u - u_0))$, so we can assume without loss of generality that a is not of this form. Then at every $\bar{y}_0$ we have $\bar{\psi}\big|_Z =$ const, $\bar{\varphi} = \bar{\varphi}(\bar{y})$, and $\bar{\psi} = \bar{\psi}(\bar{y})$. Thus the isomorphism $(t, y, v) \to \big(t, y, \bar{\varphi}(t, y)v + \bar{\psi}(t, y)\big)$ maps $\mathbf{B}$ to some object of $\mathcal{QPE}''_a(a)$.

The proof of the full density of $\mathcal{QPE}''_{0ca}(a)$ in $\mathcal{QPE}''_{ca}(a)$ is similar to the proof of part 10 in Theorem 4. $\square$

Proof of Theorem 6

1. $\mathcal{QPE}''$ is closed in $\overline{\mathcal{QPE}}$, and $\overline{\mathcal{SQPE}}$ is the subcategory of $\overline{\mathcal{QPE}}$. Therefore $\mathcal{SQPE}$ is closed in $\overline{\mathcal{SQPE}}$. $\square$

2. $\mathcal{SQPE}_n$ is closed in $\overline{\mathcal{SQPE}}$ for the same reason as in Part 1 of this Theorem. This implies that $\mathcal{SQPE}_n$ is closed in $\mathcal{SQPE}$.

Suppose $\mathbf{A}$ is an object of $\mathcal{SQPE}_0$, $F \colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{SQPE}$. Then $B^k(t, y, v) = A(t, y, v)\omega^k(t, x)$, where ω^k is defined as in (10). Hence ω^k is a function of (t, y), and B is a object of $\mathcal{SQPE}_0$.

Suppose $\mathbf{A}$ is an object of $\mathcal{SQPE}_b$, $F \colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{SQPE}$. From the first equation of (8) we obtain that

$$\frac{\bar{B}^{kl}}{\bar{B}^{11}}(t, y) = \frac{\sum_{i,j} \bar{b}^{ij} y_i^k y_j^l}{\sum_{i,j} \bar{b}^{ij} y_i^1 y_j^1}(x).$$

The right hand side is independent of t, so it is a function of y; denote this function by $\bar{B}'^{kl}(y)$. Then $A\bar{B}^{kl} = A'(t, y, v)\bar{B}'^{kl}(y)$, where $A' = AB^{11}$. It follows that $\mathbf{B}$ is an object of $\mathcal{SQPE}_b$, and $\mathcal{SQPE}_b$ is closed in $\mathcal{SQPE}$. $\square$

3. Let us recall that $\mathcal{SQPE}_{0n}$ is closed in $\mathcal{QPE}''_{0n}$. So it suffices to prove that any morphism in $\mathcal{QPE}''_{0n}$ is also a morphism in $\mathcal{SQPE}_{0n}$. Suppose $F \colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{QPE}''_{0n}$. Then $y_t^k(t, x) = A(t, y, v)\omega^k(t, x)$, where

$$\omega^k = -\bar{B}^k + \bar{a}\left(\sum_{i,j} \bar{b}^{ij} y_{ij}^k + 2\sum_{i,j} \bar{b}^{ij} (\ln \bar{\varphi})_j\, y_i^k + \sum_{i,j} \bar{b}^i y_i^k\right).$$

Since the left hand side of this equality is independent of v and $A \in \mathcal{A}_{nc}(Y)$, we conclude that $\omega^k = 0$. Thus F is a morphism in $\mathcal{SQPE}_{0n}$. Finally, $\mathcal{SQPE}_{0n} = \mathcal{QPE}''_{0n}$, is closed in $\mathcal{QPE}''_0$ and is fully dense in $\mathcal{QPE}''_n$. $\square$

4. $\mathcal{QPE}''_1$ is closed in $\overline{\mathcal{QPE}}$, so $\mathcal{SQPE}_1$ is closed in $\overline{\mathcal{SQPE}}$ and, consequently, is closed in $\mathcal{SQPE}_0$. $\square$

5. The proof is similar to the proof of part 1 of Theorem 5. $\square$

Proof of Theorem 7

From (8)-(9) and the fact that $\mathcal{SQPE}_b$ is closed in $\overline{\mathcal{SQPE}}$ it follows that the map $(t,x,u) \mapsto (t,y,\varphi u+\psi)$ is a morphism in $\mathcal{SQPE}$ with the source from $\mathcal{AQPE}$ if and only if the following conditions are satisfied:

$$\begin{cases} A(t,y,v) & = a(x,u)\bar{a}(t,x) \\ \bar{B}^{kl}(y) & = \bar{a}(t,x)\nabla y^k \nabla y^l \\ B^k(t,y,v) & = A(t,y,v)\bar{B}^k(t,y) + C^k(t,y) = \\ & = a(x,u)\left(\Delta y^k + (\eta + 2\nabla(\ln\bar{\varphi}))\nabla y^k\right) + \xi\nabla y^k \\ Q\bar{\varphi} & = \left(a\left(\Delta\bar{\varphi}+\eta\nabla\bar{\varphi}\right)+\xi\nabla\bar{\varphi}-\bar{\varphi}_t\right)v + \\ & \quad + \left(a\left(\Delta\bar{\psi}+\eta\nabla\bar{\psi}\right)+\xi\nabla\bar{\psi}-\bar{\psi}_t\right) + q\left(t,x,\bar{\varphi}v+\bar{\psi}\right) \end{cases} \tag{15}$$

1. Suppose $F\colon \mathbf{A}\to\mathbf{B}$ is a morphism in $\overline{\mathcal{AQPE}}$, $\mathbf{A}$ is an object of $\mathcal{AQPE}$. From the second equation of the system (15) it follows that $\bar{a}=\bar{a}(x)$. Using the first equation of (15) and taking into account the independence of $\bar{\varphi}$, $\bar{\psi}$ on t, we get the independence of $A=A\left(y,v\right)$ on t. It follows from the third equation of (15) that B^k is independent of t, and $B^k(y,v)=A(y,v)\bar{B}^k(t,y)+C^k\left(t,y\right)$. From this formula, by the same token as in proof of part 4 of Theorem 4 we obtain existing of functions $\mathrm{H}^k(y)$, $\Xi^k(y)$ such that $B^k=A(y,v)\mathrm{H}^k(y)+\Xi^k(y)$. Substituting $u=\bar{\varphi}(x)v+\bar{\psi}(x)$ in the last equation of (15), we obtain that Q is independent of t. This implies that target $\mathbf{B}$ of the morphism F is of the form

$$v_t = A(y,v)\left(\sum_{k,l}\bar{B}^{kl}(y)v_{kl}+\sum_k \mathrm{H}^k(y)v_k\right)+\sum_k \Xi^k(y)v_k+Q(y,v).$$

We received this form of $\mathbf{B}$ only locally. Neverytheless we can lead it to the equation of the same form but with globally defined function $A(y,v)$, for example by the way described in Remark 6. Then quadratic form $\bar{B}^{kl}$ is defined on the whole manifold Y, so we can equip Y with Riemann metric $\bar{B}^{kl}$ and finally get $\mathbf{B}\in \mathrm{Ob}_{\mathcal{AQPE}}$. ☐

2. $\mathcal{AQPE}_n = \mathcal{AQPE}\cap\mathcal{SQPE}_n$ is closed in $\mathcal{AQPE}$, because $\mathcal{SQPE}_n$ is closed in $\mathcal{SQPE}$.
Let $F\colon\mathbf{A}\to\mathbf{B}$ be a morphism in $\mathcal{SQPE}_{bn}$, and both source and target of F are objects of $\mathcal{AQPE}_n$. Then $\bar{a}$ is independent of t, and

$$a\left(x,\bar{\varphi}(t,x)v+\bar{\psi}\left(t,x\right)\right) = A\left(y(x),v\right)\bar{a}(x). \tag{16}$$

Let $x=x_0$. Suppose that the set $\left\{\left(\bar{\varphi}\left(t,x_0\right),\bar{\psi}\left(t,x_0\right)\right)\right\}$ have more than one element, and consider the intervals

$$I\left(v\right)=\left\{\left(\bar{\varphi}\left(t,x_0\right)v+\bar{\psi}\left(t,x_0\right)\right) : t\in T_{\mathbf{A}}\right\}\subseteq\mathbb{R}.$$

Then $a\left(x_0,u\right)$ is constant on any interval $u\in I\left(v\right)$, because the right hand side of (16) is independent of t. Note that $I\left(v\right)$ is a continuous function of v in the Hausdorff metric, and $\forall t$ $\bar{\varphi}\left(t,x_0\right)\neq 0$. If at any v the interval $I\left(v\right)$ does not collapses into a point, then $a\left(x_0,u\right)$ is constant on $\bigcup I\left(v\right)$. But this contradicts to the condition $a\in\mathcal{A}_{nc}\left(X\right)$. Therefore $I\left(v_0\right)$ degenerate into the point at some v_0: $\bar{\varphi}\left(t,x_0\right)v_0+\bar{\psi}\left(t,x_0\right)\equiv u_0$, and

$\bar{\varphi}v + \bar{\psi} = \bar{\varphi}(t, x_0)(v - v_0) + u_0$. By the assumption, $\operatorname{card}\left\{\left(\bar{\varphi}(t, x_0), \bar{\psi}(t, x_0)\right)\right\} > 1$, so the set $\{\bar{\varphi}(t, x_0)\}$ is nondegenerated interval. Therefore, $a(x_0, u)$ is constant on the sets $\{u < u_0\}$ and $\{u > u_0\}$. But this contradicts to the condition $a \in \mathcal{A}_{nc}(X)$ and continuity of a. This contradiction shows that for each x_0 the functions $\bar{\varphi}$, $\bar{\psi}$ are independent of t. Consequently F is a morphism in $\mathcal{AQPE}$, and $\mathcal{AQPE}_n$ is the full subcategory of $\mathcal{SQPE}_{bn}$. □

3. $\mathcal{AQPE}_0$ and $\mathcal{AQPE}_1$ are closed in $\mathcal{AQPE}$, because $\mathcal{SQPE}_0$ and $\mathcal{SQPE}_1$ are closed in $\mathcal{SQPE}$. □

4. If $a \notin \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$, then $\mathcal{AQPE}_a(a)$ is plentiful in $\mathcal{AQPE}$ by the same arguments as used in the proof of part 1 of Theorem 5, after replacement of $\bar{x}$, $\bar{y}$ to x, y respectively. □

5. Let $F\colon \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathcal{SQPE}_{na}(a)$, $\mathbf{A}$ be an object of $\mathcal{AQPE}_{na}(a)$. Then

$$a\left(\bar{\varphi}(t, x)v + \bar{\psi}(t, x)\right) = A(v)\bar{a}(x).$$

As we proved in part 2, $\bar{\varphi}, \bar{\psi}$ are independent of t, F is a morphism in $\mathcal{AQPE}$, and $\mathbf{B} \in \mathrm{Ob}_{\mathcal{AQPE}} \cap \mathrm{Ob}_{\mathcal{SQPE}_{na}(a)} = \mathrm{Ob}_{\mathcal{AQPE}_{na}(a)}$. Since $\mathcal{AQPE}_{na}(a)$ is full in $\mathcal{AQPE}_n$, we see that F is a morphism in $\mathcal{AQPE}_{na}(a)$. □

Proof of Theorem 8

1. $\mathcal{EPE}$ is closed in $\overline{\mathcal{EPE}}$, because $\mathcal{AQPE}$ is closed in $\overline{\mathcal{AQPE}}$. □

2. $\mathcal{EPE}_n$, $\mathcal{EPE}_0$, $\mathcal{EPE}_1$ are closed in $\mathcal{EPE}$, because $\mathcal{AQPE}_n$, $\mathcal{AQPE}_0$, $\mathcal{AQPE}_1$ are closed in $\mathcal{AQPE}$. □

Suppose $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{EPE}$, and $\mathbf{A} \in \mathrm{Ob}_{\mathcal{EPE}_a(a)}$. Then the first equation of (8) is of the form $A(y, u)\bar{B}^{kl}(y) = a(u)\nabla y^k \nabla y^l$. Hence $\nabla y^k \nabla y^l = g^{kl}(y)$ for some functions g^{kl}. For $\bar{B}^{kl} = g^{kl}(y)$ we get $A(y, u) = a(u)$. So $\mathbf{A}$ is the object of $\mathcal{EPE}_a(a)$, and $\mathcal{EPE}_a(a)$ is closed in $\mathcal{EPE}$. □

3. The proof is similar to the proof of Theorem 4. □

References

[Elkin, 1999] V. I. Elkin (1999), *Reduction of nonlinear control systems. A differential geometric approach*, Mathematics and its Applications, **472**, Kluwer Academic Publishers, Dordrecht.

[Hirsch, 1976] M. W. Hirsch (1967), *Differential topology*, Graduate texts in mathematics, **33**, Springer-Verlag, NY.

[Krasil'shchik, 1999] J. Krasil'shchik, A. Vinogradov (eds.) (1999), *Symmetries and conservation laws for differential equations of mathematical physics*, Translations of Mathematical Monographs, **182**, AMS, Providence, RI.

[Mac Lane, 1998] S. Mac Lane (1998), *Categories for the working mathematician*, Springer-Verlag, NY.

[Olver, 1993] P. J. Olver (1993), *Applications of Lie groups to differential equations*, Springer-Verlag, NY.

[Prokhorova, 1998] M. F. Prokhorova (1998), *Modeling of solutions of heat equation and Stefan problem with dimension decrease*, Russian Acad. Sci. Docl. Math. **58** (1), 88–90.

[Prokhorova, 2000] M. F. Prokhorova (2000), *Modeling of heat equation and Stefan problem*, Preprint VINITI N 347-V00, IMM UrBr Russian Acad. Sci., Ekaterinburg, 1–66 (in Russian).

[Prokhorova, 2001] M. F. Prokhorova (2001), *Factorization of nonlinear heat equation posed on Riemann manifold*, arXiv.org:math.AP/0108001, 1–9.